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# ROBUST DESIGN OF MULTIVARIABLE CONTROL: AN INTERVAL ARITHMETIC APPROACH

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An important aspect of analysis and design of control laws for a dynamic system is the robustness of the control laws to uncertainties in the model parameters. Interval arithmetic is used to model dynamic systems whose parameters are not known exactly, but are known to lie within an interval. Analysis techniques for interval systems are developed, and include the solution of ordinary differential equations with interval coefficients and interval initial conditions, and the solution of linear interval equations. Feedback stabilization of interval systems by state feedback using interval Routh Hurwitz arrays is developed.

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# PROJECT SUMMARY


This report studies some aspects of analysis and controller design for linear systems with parameter uncertainties. It is assumed that the uncertainties are quantified as intervals and the relevant parameter can assume any value from the interval.

Unlike conventional methods, use of *interval arithmetic* is investigated. Interval arithmetic provides a mechanism for dealing with the uncertainties in a very general framework. All the calculations have to be performed in terms of intervals instead of real and/or complex numbers.

Historically, the use of interval arithmetic was motivated by an effort to find bounds on round-off errors in numerical computations, and therefore, typically the intervals used were of the order of the last significant place after the decimal on the computer on which it was being employed. The premises in this report is that by increasing the interval size to accommodate model uncertainty, if a controller is designed for an interval instead of a particular value from that interval, stability of the system in face of complete variation in corresponding parameter is automatically guaranteed!

At the current time the above objective is achieved only at the cost of overbounding the uncertainties hence the controller, when it exists, will be conservative.

# 1. INTRODUCTION AND OVERVIEW

## 1.1. INTRODUCTION

The fields of optimization and control theory are well developed disciplines with very strong theoretical and methodological foundations. Although a significant effort has been underway to develop both of the above areas independently, a large variety of current problems require us to pool the resources from these and other related areas and to put forth a combined effort to solve them. Among the problems presenting these challenges is the development of a symbiotic relationship between system theory principles and practical problems such as design of flight control systems, communication systems, transportation systems, *etc.*

Control theory has been a subject of intensive research for several decades. However, until recently, the principle effort of research was focussed on classical design methodologies such as Bode diagram, root locus, Nyquist plots, *etc.* These design methods are extremely effective for solving small scale and relatively less complicated problems. However, they have limited utility in modern multivariable control systems analysis and design due to the stringent design and performance requirements commanded by current technology.

An important aspect of analysis and design of a system is to make it "robust" in face of uncertainties in the model parameters and/or operating conditions [3], [5], [7], [9], [12], [16], [35], [40] and [44]. The need to incorporate robustness in design

is necessitated by the fact that for most practical systems, the model is known only approximately. For example, in the aircraft industry, the aircraft model is constructed using the data obtained from the wind-tunnel experiments on the aircraft body. As a consequence, the parameters of the model would not have a specific value, rather they are known to lie within an interval. Since the actual flight data are not available, the controller should be able to account for the unmodeled parameters that can be obtained only when the aircraft is airborne. Design of a "robust" controller therefore becomes a priority in such applications.

Fortunately, in the last decade the momentum has visibly shifted to integrating more sophisticated mathematical theories to solve these problems. As a consequence, the newer design techniques are capable of meeting the required performance measures and at the same time giving the designer sufficient freedom to incorporate additional "desirable features" of robustness. Several theories, notably those developed in [3], [9], [11], [13], [14], [19], [24], [33] and [44] have been put forth that enable us to design robust systems. One assumption underlying all of these theories is that the *nominal* model of the system is known. The design is carried out based on this model, and the final product is analyzed for the degree of robustness.

The approach adopted in the proposed research is significantly different from the conventional approaches to the solution of this problem. In this project, we will investigate the application of *interval mathematics* in analysis and design of multivariable control systems. Interval analysis, as introduced by Moore [26], [27] and extended by several researchers *e.g.*, [1], [30], considers each number as an interval instead of a fixed point in the complex plane. Although the original motivation behind the use

of interval analysis was to capture all rounding-errors during computations, the interval arithmetic was developed as a parallel to conventional arithmetic. Since, as discussed, the parameters of the models in modern multivariable systems are usually known to be within an interval, the use of interval arithmetic for their analysis and design appears to be a logical direction.

The prediction and control of dynamical processes described by general physical laws or experimental data is a basic problem in engineering. Therefore, the starting point in analysis and design of a practical system is to obtain an accurate description or a model. These models are usually obtained by means of simulations and/or laboratory experiments. If the model parameters are known precisely, the solution of this representation problem is very well understood. In practice, the real conditions under which the system would operate can only be approximated in the laboratory experiments. Hence it is reasonable to assume that a precise knowledge of the process or model parameters is *atypical*. This is especially true in the case of the aircraft industry where wind-tunnel tests are used to obtain the best possible approximation of the "true" model of the aircraft. The control elements such as rudder, elevator, *etc.*, are installed based on the available model.

For applications such as the one described above, although a precise model is almost never available, the extensive experience with such experiments over the past several decades does enable the engineers to specify the degree of uncertainty associated with various parameters that determine the performance of given system.

In classical frequency domain techniques for single-input, single-output systems, the concept of gain and phase margins are well understood. These margins provide

effective measure in determining the robustness of the given plant. These notions, however, do not directly extend to multi-input, multi-output plants. In fact, it has been shown for the multivariable case that even if each channel in the feedback mechanism possesses some desired gain and phase margins, a small perturbation in the system can result in closed-loop instability.

Analysis and design of a system whose parameters are known to lie within a range (rather than having an exact value) has received considerable attention over the last decade. Various analysis and design methodologies used by control systems engineers are essentially meant for application to a "nominal" model. The resulting design is said to be robust if the system performs within acceptable limits in the face of significant parameters variations and model uncertainties.

This situation has led to fundamental extensions and re-evaluations of the modeling philosophy, an extensive and rigorous development of powerful estimation techniques and the advent of various approaches to the design of controllers that are robust against such uncertainties. Some of the more important of the advances that address the design of robust controllers are (1) *Robust Servomechanism Control*, where a good understanding of the nature of uncertainties associated with the system and the environment in which it operates enables us to design servomechanism controllers that guarantee asymptotic regulation, (2) *Linear Quadratic Regulator* with multiple loop state feedback. These regulators have excellent robustness properties when measured by classical criteria of gain and phase margins and can undergo substantial gain and phase perturbations without becoming unstable, and (3) *Optimal Hankel Norm Optimization* based techniques that use matrix valued interpolation theory to obtain

robust controllers for the given plant.

In practically all of the research that has been conducted in robust controller design, the underlying model is assumed to have some nominal values of the parameters. This choice of parameters is carried through the design. Once a controller is found, the robustness of the resulting controller is determined from the final design. Intuitively, it appears more appealing to perform the complete design using intervals in which these parameters lie, instead of selecting one particular value from these intervals. This leads to the natural choice of *interval arithmetic* as the most logical tool to address the problem of robust controller design. The subject of interval analysis was developed in mid-sixties in quest for rigor in numerical computations on computers using finite precision arithmetic. Interval arithmetic treats intervals as a new kind of number. Computations in appropriately rounded arithmetic produce results that contain both ordinary machine arithmetic results as well as infinite precision arithmetic results.

The use of interval arithmetic was motivated by an effort to find bounds on round-off errors in numerical computations, and therefore, typically the intervals used were of the order of the last significant place after the decimal on the computer on which it was being employed. However, since the underlying principles for various operations such as addition and multiplication of the intervals have already been established, it remains to increase the interval size that suits the practical problem of enclosing the uncertainty in the model parameters, and extend the methods of analysis and design of systems using interval arithmetic. It is not too difficult to visualize that if a controller is designed for an interval instead of a particular value from that interval,

stability of the system in face of complete variation in corresponding parameter is automatically guaranteed! It should be emphasized that this is achieved only at the cost of overbounding the uncertainties hence the controller, when it exists, will be conservative.

In the rest of this report, we will consider system described by their state-space representation:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (1.1a)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \quad (1.1b)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$ and $\mathbf{y}(t) \in \mathbb{R}^p$. Or in their transfer function representation:

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s) \quad (1.2a)$$

$$= \frac{\mathbf{N}(s)}{d(s)}\mathbf{U}(s) \quad (1.2b)$$

$$= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \quad (1.2c)$$

In most existing analysis and design techniques, it is assumed that the matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ (describing the various parameters of the system) are known precisely. However, as mentioned in the previous section, for most practical systems, this is seldom the case. Therefore, it will be assumed that $\mathbf{A} \in \mathbb{IR}^{n\times n}$, $\mathbf{B} \in \mathbb{IR}^{n\times m}$ and $\mathbf{C} \in \mathbb{IR}^{p\times n}$, where the notation $\mathbb{IR}$ stands for an interval over the field of real numbers. Accordingly $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are matrices whose elements (some or all) are intervals over the field of real numbers. In a similar manner, it will be assumed that the coefficients of various polynomials in the transfer function (matrix) are intervals.

## 1.2. Layout of the Report

The remainder of this report is organized as follows:

1. **REVIEW OF INTERVAL ARITHMETIC TECHNIQUES:** Although interval arithmetic was conceived roughly 20 years ago, its use has been restricted to the numerical analysis community. Therefore to make the project self-contained, a sufficiently detailed review was conducted.

2. **ANALYSIS OF MULTIVARIABLE SYSTEMS:** Frequently, a good understanding and knowledge of the system under consideration enables one to design a better controller. Hence various analysis techniques using interval arithmetic were developed. These analysis techniques are (a) the solution of systems of differential equations with interval coefficients and/or interval initial conditions, (b) and the solution of linear interval equations.

3. **DESIGN OF ROBUST CONTROLLERS:** Since the underlying principles of interval analysis implicitly account for a range of numbers rather than a particular number, it is easy to see that the design, when it exists, will automatically be robust *albeit* with some conservatism. In particular, the problems of feedback stabilization by means of state feedback and simultaneous stabilization for single input single output systems were addressed.

# 2. REVIEW OF INTERVAL ARITHMETIC

## 2.1. Notation

Unless stated otherwise, we will use the following notation in the rest of this report:

| | |
|---|---|
| $[\,\check{a},\ \hat{a}\,]$ | interval scalars |
| $\check{a},\ \hat{a}$ | lower and upper limits of interval scalar |
| $[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$ | interval matrix |
| $\check{\mathbf{A}},\ \hat{\mathbf{A}}$ | lower and upper limits of interval matrices |
| $[\,\check{\mathbf{a}},\ \hat{\mathbf{a}}\,]$ | interval vector |
| $\check{\mathbf{a}},\ \hat{\mathbf{a}}$ | lower and upper limits of interval vectors |
| $x$ | a point from the interval $[\,\check{\mathbf{x}},\ \hat{\mathbf{x}}\,]$ |
| $\mathbf{x}$ | a point vector from the interval vector $[\,\check{\mathbf{x}},\ \hat{\mathbf{x}}\,]$ |
| $\mathbf{X}$ | a point matrix from the interval matrix $[\,\check{\mathbf{X}},\ \hat{\mathbf{X}}\,]$ |
| $\mathbb{R}$ | field of real numbers |
| $\mathbb{IR}$ | interval scalar assuming value in $\mathbb{R}$ |
| $\mathbb{IR}^n$ | interval vector assuming value in $\mathbb{R}^n$ |
| $\mathbb{IR}^{n\times m}$ | interval matrix assuming value in $\mathbb{R}^{n\times m}$ |
| $[\,\check{a},\ \hat{a}\,]\vee[\,\check{b},\ \hat{b}\,]$ | joint of $[\,\check{a},\ \hat{a}\,]$ and $[\,\check{b},\ \hat{b}\,] = [\min\{\check{a},\check{b}\},\max\{\hat{a},\hat{b}\}]$ |
| $\mathbf{w}([\,\check{A},\ \hat{A}\,])$ | width of $[\,\check{A},\ \hat{A}\,] = \hat{A} - \check{A}$ |
| $\mathbf{A}\otimes\mathbf{B}$ | Kronecker product of two matrices defined in eq. (4.2) |
| $\mathrm{vec}\,(\mathbf{A})$ | a vector of the columns of a matrix defined in eq. (4.3) |

## 2.2. Interval Scalars & Functions

Let $\mathbb{I}$ be the set of real compact intervals $[\,\check{a},\ \hat{a}\,]$, $\check{a}, \hat{a} \in \mathbb{R}$. General interval operations can be defined as [1], [27]:

**Definition 2.1**: Let $*$ be a binary operation on the set of real numbers $\mathbb{R}$. If $[\,\check{a},\ \hat{a}\,]$, $[\,\check{b},\ \hat{b}\,] \in \mathbb{I}\mathbb{R}$, then

$$[\,\check{a},\ \hat{a}\,] * [\,\check{b},\ \hat{b}\,] \;=\; \{a * b \;:\; a \in [\,\check{a},\ \hat{a}\,],\ b \in [\,\check{b},\ \hat{b}\,]\}, \quad [\,\check{a},\ \hat{a}\,], [\,\check{b},\ \hat{b}\,] \in \mathbb{I}, \tag{2.1}$$

where $*$ stands for $+$, $-$, $\cdot$ and $/$. Further the operator $/$ is only defined for the operations $[\,\check{a},\ \hat{a}\,]/[\,\check{b},\ \hat{b}\,]$, $0 \notin [\,\check{b},\ \hat{b}\,]$.

It is clear from the above definition that

$$a * b \;\in\; [\,\check{a},\ \hat{a}\,] * [\,\check{b},\ \hat{b}\,] \tag{2.2}$$

defined as the *inclusion principle of interval arithmetic*. It can be interpreted to mean that the sum, difference, product and quotient of the reals (possibly unknown) are contained in the sum, difference, product and quotient of the including intervals which are known precisely.

The real numbers can be defined as point intervals,

$$a \;=\; [\,\check{a},\ \hat{a}\,] \;=\; [\,\hat{a},\ \check{a}\,] \;=\; [\,a,\ a\,]. \tag{2.3}$$

With the above definition of point intervals, we can define the operations between point intervals and intervals with finite width. Further, the usual arithmetic operations are carried over to intervals as follows:

**Definition 2.2**: Let $[\check{a},\ \hat{a}]$, $[\check{b},\ \hat{b}] \in \mathbb{IIR}$, then

$$
\begin{aligned}
[\check{a},\ \hat{a}] + [\check{b},\ \hat{b}] &= [\check{a}+\check{b},\ \hat{a}+\hat{b}] \\
[\check{a},\ \hat{a}] - [\check{b},\ \hat{b}] &= [\check{a}-\hat{b},\ \hat{a}-\check{b}] \\
[\check{a},\ \hat{a}] \cdot [\check{b},\ \hat{b}] &= [\min(\check{a}\check{b}, \hat{a}\check{b}, \check{a}\hat{b}, \hat{a}\hat{b}), \max(\check{a}\check{b}, \hat{a}\check{b}, \check{a}\hat{b}, \hat{a}\hat{b})] \\
[\check{a},\ \hat{a}]/[\check{b},\ \hat{b}] &= [\check{a},\ \hat{a}] \cdot \left[\frac{1}{\hat{b}}, \frac{1}{\check{b}}\right] \quad 0 \notin [\check{b},\ \hat{b}].
\end{aligned}
\tag{2.4}
$$

Equation (2.4) clearly indicates that subtraction and division in interval arithmetic are not the inverse operations of addition and multiplication. In fact this is one of the main properties of interval arithmetic that distinguishes it from the real arithmetic. Some of the properties of interval operations that will be useful in the sequel are:

**Definition 2.3**: Let $[\check{a},\ \hat{a}]$, $[\check{b},\ \hat{b}]$, $[\check{c},\ \hat{c}] \in \mathbb{IIR}$, then

$$
\begin{aligned}
&[\check{a},\ \hat{a}] + [\check{b},\ \hat{b}] = [\check{b},\ \hat{b}] + [\check{a},\ \hat{a}] && \text{(commutativity)} \\
&[\check{a},\ \hat{a}] + ([\check{b},\ \hat{b}] + [\check{c},\ \hat{c}]) = ([\check{a},\ \hat{a}] + [\check{b},\ \hat{b}]) + [\check{c},\ \hat{c}] && \text{(associativity)} \\
&[\check{a},\ \hat{a}] \cdot [\check{b},\ \hat{b}] = [\check{b},\ \hat{b}] \cdot [\check{a},\ \hat{a}] && \text{(commutativity)} \\
&[\check{a},\ \hat{a}]([\check{b},\ \hat{b}] \cdot [\check{c},\ \hat{c}]) = ([\check{a},\ \hat{a}] \cdot [\check{b},\ \hat{b}])[\check{c},\ \hat{c}] && \text{(associativity)} \\
&[\check{a},\ \hat{a}]([\check{b},\ \hat{b}] + [\check{c},\ \hat{c}]) \subseteq [\check{a},\ \hat{a}][\check{b},\ \hat{b}] + [\check{a},\ \hat{a}][\check{c},\ \hat{c}] && \text{(subdistributivity)} \\
&a([\check{b},\ \hat{b}] + [\check{c},\ \hat{c}]) = a[\check{b},\ \hat{b}] + a[\check{c},\ \hat{c}] && a \in \mathbb{R} \\
&[\check{a},\ \hat{a}] \subseteq [\check{b},\ \hat{b}], [\check{c},\ \hat{c}] \subseteq [\check{d},\ \hat{d}] \Rightarrow [\check{a},\ \hat{a}] * [\check{c},\ \hat{c}] \subseteq [\check{b},\ \hat{b}][\check{d},\ \hat{d}].
\end{aligned}
\tag{2.5}
$$

The last property is called *inclusion isotonicity* of interval operations.

Note that all these operations reduce to real arithmetic when the intervals are degenerate intervals or point intervals *i.e.*, $\check{a} = \hat{a}$ and $\check{b} = \hat{b}$. In particular in the division operation with point intervals, $0 \in [\check{b},\ \hat{b}]$ if and only if $\check{b} = \hat{b} = 0$.

The operations defined by Definition 2.4 are the binary operation on the operators. In addition to the binary operations, there are operations that are associated with a single interval, called *unary* operations, characterized next.

**Definition 2.4**: If $r(x)$ is a unary operation on $\mathbb{R}$, then

$$r([\,\check{x},\ \hat{x}\,]) \;=\; \left[\begin{matrix}\min r(x) \\ x \in [\,\check{x},\ \hat{x}\,]\end{matrix},\ \begin{matrix}\max r(x) \\ x \in [\,\check{x},\ \hat{x}\,]\end{matrix}\right] \tag{2.6}$$

defines a (subordinate) unary operation on $\mathbb{IR}$.

For example, the above definition accounts for operations like $[\,\check{x},\ \hat{x}\,]^k$, $k \in \mathbb{R}$, $e^{[\,\check{x},\ \hat{x}\,]}$, $\ln([\,\check{x},\ \hat{x}\,])$, $\sin([\,\check{x},\ \hat{x}\,])$, $\cos([\,\check{x},\ \hat{x}\,])$, etc.

An important property of interval computation is the ***interval monotonicity*** defined as follows:

**Definition 2.5**: Let $[\,\check{a},\ \hat{a}\,]^{(k)}$, $[\,\check{a},\ \hat{a}\,]^{(k)} \in \mathbb{IR}$, $k = 1,2$ and assume that

$$[\,\check{a},\ \hat{a}\,]^{(k)} \;\subseteq\; [\,\check{b},\ \hat{b}\,]^{(k)}, \quad k = 1,2.$$

Then for the binary operations $* \in \{+,-,\cdot,/\}$,

$$[\,\check{a},\ \hat{a}\,]^{(1)} * [\,\check{a},\ \hat{a}\,]^{(2)} \;\subseteq\; [\,\check{b},\ \hat{b}\,]^{(1)} * [\,\check{b},\ \hat{b}\,]^{(2)}. \tag{2.7}$$

**Definition 2.6**: The distance between two intervals $[\,\check{a},\ \hat{a}\,]$ and $[\,\check{b},\ \hat{b}\,] \in \mathbb{IR}$ is defined as

$$d([\,\check{a},\ \hat{a}\,],[\,\check{b},\ \hat{b}\,]) \;=\; \max\{|\check{a}-\check{b}|,\ |\hat{a}-\hat{b}|\}. \tag{2.8}$$

It can be proved that the map $d$ introduces a *metric* in $\mathbb{IR}$. Note that

$$d([\,\check{a},\ \hat{a}\,],[\,\check{b},\ \hat{b}\,]) \geq 0 \qquad \text{and} \qquad d([\,\check{a},\ \hat{a}\,],[\,\check{b},\ \hat{b}\,]) = 0 \Leftrightarrow [\,\check{a},\ \hat{a}\,] = [\,\check{b},\ \hat{b}\,]$$

$$d([\,\check{a},\ \hat{a}\,],[\,\check{b},\ \hat{b}\,]) \leq d([\,\check{a},\ \hat{a}\,],[\,\check{c},\ \hat{c}\,]) + d([\,\check{b},\ \hat{b}\,],[\,\check{c},\ \hat{c}\,]). \tag{2.9}$$

Notice that (2.9) is the triangle inequality and may be verified as follows:

$$\begin{aligned} d([\,\check{a},\ \hat{a}\,],[\,\check{c},\ \hat{c}\,]) + d([\,\check{b},\ \hat{b}\,],[\,\check{c},\ \hat{c}\,]) &= \max\{|\check{a}-\check{c}|,\ |\hat{a}-\hat{c}|\} + \max\{|\check{b}-\check{c}|,\ |\hat{b}-\hat{c}|\} \\ &\geq \max\{|\check{a}-\check{c}| + |\check{b}-\check{c}|,\ |\hat{a}-\hat{c}| + |\hat{b}-\hat{c}|\} \\ &\geq \max\{|\check{a}-\check{b}|,\ |\hat{a}-\hat{b}|\} \\ &= d([\,\check{a},\ \hat{a}\,],[\,\check{b},\ \hat{b}\,]). \qquad (2.10) \end{aligned}$$

The metric for real intervals defined above is the Hausdorff metric on IIR and is a generalization of the distance between two points in a metric space. Further, it has been shown that the metric space is complete. Existence of metric in IIR makes it a topological space and therefore the concepts of *convergence* and *continuity* may be used in the usual manner. In particular, a sequence of intervals $\{[\,\check{a},\ \hat{a}\,]^{(k)}\}$, $k = 0, 1, \ldots, \infty$, converges to an interval $[\,\check{a},\ \hat{a}\,]$ if and only if the sequence of the bounds of the individual members of the sequence converge to the corresponding bounds $\check{a}$ and $\hat{a}$ *i.e.*

$$\lim_{k\to\infty} [\,\check{a},\ \hat{a}\,]^{(k)} = [\,\check{a},\ \hat{a}\,] \ \Leftrightarrow\ \left( \lim_{k\to\infty} \check{a}^{(k)} = \check{a} \quad \text{and} \quad \lim_{k\to\infty} \hat{a}^{(k)} = \hat{a} \right). \qquad (2.11)$$

**Theorem 2.7**: Given a sequence of intervals $\{[\,\check{a},\ \hat{a}\,]^{(k)}\}_{k=0}^{\infty}$, assume that the condition

$$[\,\check{a},\ \hat{a}\,]^{(0)} \supseteq [\,\check{a},\ \hat{a}\,]^{(1)} \supseteq [\,\check{a},\ \hat{a}\,]^{(2)} \supseteq \cdots \qquad (2.12)$$

is satisfied. Then, the sequence of intervals $\{[\,\check{a},\ \hat{a}\,]^{(k)}\}_{k=0}^{\infty}$ converges to the interval $[\,\check{a},\ \hat{a}\,] = \cap_{k=0}^{\infty}[\,\check{a},\ \hat{a}\,]^{(k)}$. ■

The above results immediately leads to very useful notions associated with functions in interval arithmetic. They are stated without proof.

**Theorem 2.8**: The operations $* \in \{+, -, \cdot, /\}$ between intervals are continuous. ■

**Theorem 2.9**: Let $f$ be a continuous function and let

$$f([\ \check{x},\ \hat{x}\ ]) = [\ \min_{x \in [\ \check{x},\ \hat{x}\ ]} f(x),\ \max_{x \in [\ \check{x},\ \hat{x}\ ]} f(x)\ ], \tag{2.13}$$

then $f([\ \check{x},\ \hat{x}\ ])$ is a continuous interval expression. ■

## 2.3. INTERVAL MATRICES

Some of the results of point matrices can be easily extended to interval matrices. However, due to the nature of interval operations, several of them cannot be extended directly to interval matrices. In next few paragraphs, we discuss some properties of interval matrices and operations with interval matrices that will be of use in the sequel.

**Definition 2.10**: Two interval matrices $[\ \check{\mathbf{A}},\ \hat{\mathbf{A}}\ ]$ and $[\ \check{\mathbf{B}},\ \hat{\mathbf{B}}\ ] \in \mathbb{IIR}^{n \times m}$ are equal, *i.e.*, $[\ \check{\mathbf{A}},\ \hat{\mathbf{A}}\ ] = [\ \check{\mathbf{B}},\ \hat{\mathbf{B}}\ ]$ if and only if every element of the two matrices are equal *i.e.*, $[\ \check{a}_{ij},\ \hat{a}_{ij}\ ] = [\ \check{b}_{ij},\ \hat{b}_{ij}\ ]$.

**Definition 2.11**: Given $[\ \check{\mathbf{A}},\ \hat{\mathbf{A}}\ ]$ and $[\ \check{\mathbf{B}},\ \hat{\mathbf{B}}\ ] \in \mathbb{IIR}^{n \times m}$, then

$$[\ \check{\mathbf{A}},\ \hat{\mathbf{A}}\ ]\ \subseteq\ [\ \check{\mathbf{B}},\ \hat{\mathbf{B}}\ ]\quad \Leftrightarrow\ [\ \check{a}_{ij},\ \hat{a}_{ij}\ ]\ \subseteq\ [\ \check{b}_{ij},\ \hat{b}_{ij}\ ], \tag{2.14}$$

$i\ =\ 1, 2, \ldots, n$ and $j\ =\ 1, 2, \ldots, m$.

The general operations between interval matrices can be defined as:

**Definition 2.12**: ADDITION, SUBTRACTION – Given $[\ \check{\mathbf{A}},\ \hat{\mathbf{A}}\ ]$ and $[\ \check{\mathbf{B}},\ \hat{\mathbf{B}}\ ] \in \mathbb{IIR}^{n \times m}$, with elements $[\ \check{a}_{ij},\ \hat{a}_{ij}\ ]$ and $[\ \check{b}_{ij},\ \hat{b}_{ij}\ ]$ respectively, then

$$[\ \check{\mathbf{A}},\ \hat{\mathbf{A}}\ ]\ \pm\ [\ \check{\mathbf{B}},\ \hat{\mathbf{B}}\ ] = \Big( [\ \check{a}_{ij},\ \hat{a}_{ij}\ ] \pm [\ \check{b}_{ij},\ \hat{b}_{ij}\ ] \Big) \tag{2.15}$$

where the right hand side denotes the $(i,j)$-th element of the sum or difference of the two matrices on the left hand side.

MULTIPLICATION - Let $[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,] \in \mathbb{IR}^{n\times r}$ and $[\,\check{\mathbf{B}},\ \hat{\mathbf{B}}\,] \in \mathbb{IR}^{r\times m}$, with elements $[\,\check{a}_{ij},\ \hat{a}_{ij}\,]$ and $[\,\check{b}_{ij},\ \hat{b}_{ij}\,]$ respectively, then

$$[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,][\,\check{\mathbf{B}},\ \hat{\mathbf{B}}\,] = \left(\sum_{k=1}^{r}[\,\check{a}_{ik},\ \hat{a}_{ik}\,][\,\check{b}_{rj},\ \hat{b}_{rj}\,]\right) \tag{2.16}$$

defines the product of two interval matrices. Interval matrix and interval vector product can be defined in an identical manner.

The next theorem outlines some useful properties of interval matrices:

**Theorem 2.13**: Let $[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$, $[\,\check{\mathbf{B}},\ \hat{\mathbf{B}}\,]$ and $[\,\check{\mathbf{C}},\ \hat{\mathbf{C}}\,]$ be interval matrices of appropriate dimensions, then

$$[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,] + [\,\check{\mathbf{B}},\ \hat{\mathbf{B}}\,] = [\,\check{\mathbf{B}},\ \hat{\mathbf{B}}\,] + [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,] \qquad \text{(commutativity)}$$

$$[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,] + ([\,\check{\mathbf{B}},\ \hat{\mathbf{B}}\,] + [\,\check{\mathbf{C}},\ \hat{\mathbf{C}}\,]) = ([\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,] + [\,\check{\mathbf{B}},\ \hat{\mathbf{B}}\,]) + [\,\check{\mathbf{C}},\ \hat{\mathbf{C}}\,]$$

(associativity)

$$([\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,] + [\,\check{\mathbf{B}},\ \hat{\mathbf{B}}\,])[\,\check{\mathbf{C}},\ \hat{\mathbf{C}}\,] \subseteq [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,][\,\check{\mathbf{C}},\ \hat{\mathbf{C}}\,] + [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,][\,\check{\mathbf{C}},\ \hat{\mathbf{C}}\,]$$

(subdistributivity)

$$[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,] + \mathbf{0} = [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,] \qquad \mathbf{0} \text{ is a null matrix}$$

$$[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]\mathbf{I} = [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,] \qquad \mathbf{I} \text{ is an identity matrix}$$

$$\mathbf{A}([\,\check{\mathbf{B}},\ \hat{\mathbf{B}}\,] + [\,\check{\mathbf{C}},\ \hat{\mathbf{C}}\,]) = \mathbf{A}[\,\check{\mathbf{B}},\ \hat{\mathbf{B}}\,] + \mathbf{A}[\,\check{\mathbf{C}},\ \hat{\mathbf{C}}\,] \qquad \mathbf{A} \text{ a nominal matrix}$$

■

# 3. SOLUTION OF INTERVAL DIFFERENTIAL EQUATIONS

The solution of high order ordinary differential equations (ODE) is a standard component of all control design and analysis techniques and extremely efficient techniques exist for the solution of the same [7], [19]. However, frequently there is some uncertainty in the exact knowledge of the initial conditions. This in conjunction with uncertainty in the parameter values can lead to misinterpretation of the actual solutions. With that in mind, in this section, we present the solution of the differential equations with interval coefficients and interval initial conditions. The aim of this section is to estimate the inclusion of the solution set for such differential equations. It should be clearly stated here that the rest of this section has been adopted from a paper entitled "An Interval Method for Systems of ODE" that appeared in the Proceedings of 1985 International Symposium on Interval Mathematics. Since the result is an extremely important one and not easily accessible, it has been rewritten (with some modifications) and included in this report.

Consider the following system of $n$ ODE's:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\,(t, [\,\check{\mathbf{c}},\ \hat{\mathbf{c}}\,], \mathbf{x}(t)), \quad \mathbf{x}(t_0) = [\,\check{\mathbf{x}}_0,\ \hat{\mathbf{x}}_0\,] \tag{3.1}$$

where $\mathbf{c} \in [\mathbf{c}] = [\,\check{\mathbf{c}},\ \hat{\mathbf{c}}\,] \in \mathbb{I}\mathbb{R}^{\ell}$ and $\mathbf{x}_0 \in [\mathbf{x}_0] = [\,\check{\mathbf{x}}_0,\ \hat{\mathbf{x}}_0\,] \in \mathbb{I}\mathbb{R}^n$ are the system parameters and initial conditions respectively. Note that the uncertainty in the parameters

as well as that in the initial conditions is assumed to lie within the corresponding interval vectors.

Let the time interval on which we wish to determine the solution of the above problem be denoted by $[\,\check{t},\ \hat{t}\,] = [t_o, t_f]$. Then, the problem that we address is that of determining an enclosure $[\,\check{\mathbf{s}},\ \hat{\mathbf{s}}\,]$ of the set $\{\mathbf{x}(t)\}$ of all solutions of (3.1) (assuming that the solutions exist) on the desired time interval. Equivalently, we determine $[\,\check{\mathbf{s}},\ \hat{\mathbf{s}}\,]$ such that

$$\check{\mathbf{s}}(t) \leq \mathbf{x}(t) \leq \hat{\mathbf{s}}(t) \quad \forall\ \mathbf{x} \tag{3.2}$$

for each $t \in [t_o, t_f]$. Let $\mathbf{f}$ define a mapping

$$\mathbf{f} : \mathbf{f} \rightarrow [\,\check{t},\ \hat{t}\,] \times [\,\check{\mathbf{c}},\ \hat{\mathbf{c}}\,] \times [\,\check{\mathbf{d}},\ \hat{\mathbf{d}}\,], \quad [\,\check{\mathbf{d}},\ \hat{\mathbf{d}}\,] \in \mathrm{I\!I\!R}^n, \tag{3.3}$$

where $[\,\check{\mathbf{d}},\ \hat{\mathbf{d}}\,]$ is the domain of $\mathbf{x}$. We impose the following conditions on the function:

1. $\mathbf{f}(\cdot)$ is continuous with respect to $\mathbf{c}$
2. Lipschitzian† with respect to $\mathbf{x}$ and $t$
3. and isotone with respect to $\mathbf{x}$.

Then, the following algorithm will compute the required enclosures.

**Algorithm ODE:** *Compute Enclosure of a System of ODE*

Let $h > 0$ be a sufficiently small step such that $t_k = t_o + kh \in [\,\check{t},\ \hat{t}\,]$, where $k$ is an integer and let $k = 0, 1, \ldots, \hat{k}$.

Let $\check{\mathbf{s}}(t_o) = \check{\mathbf{x}}_o$ and $\hat{\mathbf{s}}(t_o) = \hat{\mathbf{x}}_o$;

Let $\hat{F}_i([\bar{T}], [\bar{D}]) = \{\hat{f}_i(t, x) : t \in [\bar{T}], \mathbf{x} \in [\bar{D}]\}$, $\check{F}_i([\bar{T}], [\bar{D}]) = \{\check{f}_i(t, x) : t \in [\bar{T}], \mathbf{x} \in [\bar{D}]\}$, $i = 1, 2, \ldots, n$ for each $[\bar{T}] \subset [\,\check{t},\ \hat{t}\,]$ and $[\bar{D}] \subset [\,\check{\mathbf{d}},\ \hat{\mathbf{d}}\,]$.

† A function satisfying Lipschitz condition: $|f(x, y) - f(\hat{x}, y)| \leq \ell |x - \hat{x}|$, is said to be Lischitzian.

Assume further that we have already computed $\check{\mathbf{s}}(t_k)$ and $\hat{\mathbf{s}}(t_k)$, such that $\check{s}_i(t_k) \leq \bar{\mathbf{x}}_i(t_k) \leq \hat{s}_i(t_k)$, $i = 1, 2, \ldots, n$. Then the following recursion computes the vectors

$\check{\mathbf{s}}(t) = [\check{s}_1 \;\; \check{s}_2 \;\; \cdots \;\; \check{s}_n]^T$ and

$\hat{\mathbf{s}}(t) = [\hat{s}_1 \;\; \hat{s}_2 \;\; \cdots \;\; \hat{s}_n]^T$,

in the interval $[t_k, t_{k+1}]$.

**upper bound** $\hat{\mathbf{s}}(t)$:

$\hat{z}_i^{(o)} = [\check{d}_i, \hat{d}_i]$, $i = 1, 2, \ldots, n$

**for** $r = 0, 1, \ldots, \bar{r}$,

$$\begin{aligned}
\left[\hat{p}_i^{(r)}, \hat{q}_i^{(r)}\right] &= \hat{F}_i\left(t_k, \hat{z}_1^{(r)}, \hat{z}_2^{(r)}, \ldots, \hat{z}_n^{(r)}\right), \; i = 1, 2, \ldots, n \\
\hat{z}_i^{(r+1)} &= \hat{s}_i(t_k) \vee \left(\hat{s}_i(t_k) + \hat{p}_i^{(r)} h\right) \vee \left(\hat{s}_i(t_k) + \hat{q}_i^{(r)} h\right), \; i = 1, 2, \ldots, n \\
\hat{s}_i(t) &= \hat{s}_i(t_k) + (t - t_k)\hat{q}_i^{(\bar{r})}, \; i = 1, 2, \ldots, n, \; t \in [t_k, t_{k+1}]
\end{aligned}$$

**end**

**lower bound** $\check{\mathbf{s}}(t)$:

$\check{z}_i^{(o)} = [\check{d}_i, \hat{d}_i]$, $i = 1, 2, \ldots, n$

**for** $r = 0, 1, \ldots, \bar{r}$,

$$\begin{aligned}
\left[\check{p}_i^{(r)}, \check{q}_i^{(r)}\right] &= \check{F}_i\left(t_k, \check{z}_1^{(r)}, \check{z}_2^{(r)}, \ldots, \check{z}_n^{(r)}\right), \; i = 1, 2, \ldots, n \\
\check{z}_i^{(r+1)} &= \check{s}_i(t_k) \vee \left(\check{s}_i(t_k) + \check{p}_i^{(r)} h\right) \vee \left(\check{s}_i(t_k) + \check{q}_i^{(r)} h\right), \; i = 1, 2, \ldots, n \\
\check{s}_i(t) &= \check{s}_i(t_k) + (t - t_k)\check{p}_i^{(\bar{r})}, \; i = 1, 2, \ldots, n, \; t \in [t_k, t_{k+1}]
\end{aligned}$$

**end.**

Effectively, the algorithm is computing the following quantities:

1. the intervals $f_i(t, [\,\check{\mathbf{c}}, \hat{\mathbf{c}}\,], \mathbf{x}(t)) = \{f_i(t, \mathbf{c}, \mathbf{x}(t)), \mathbf{c} \in [\,\check{\mathbf{c}}, \hat{\mathbf{c}}\,]\}$, $i = 1, 2, \ldots, n$, for each value of $t \in [\,\check{t}, \hat{t}\,]$ and the vector $\mathbf{x}(t) \in [\,\check{\mathbf{d}}, \hat{\mathbf{d}}\,]$. The functions

$\hat{F}_i\left(t_k, \hat{z}_1^{(r)}, \hat{z}_2^{(r)}, \ldots, \hat{z}_n^{(r)}\right)$, and $\check{F}_i\left(t_k, \check{z}_1^{(r)}, \check{z}_2^{(r)}, \ldots, \check{z}_n^{(r)}\right)$, $i = 1, 2, \ldots, n$, respectively, denote the end points of each of the interval.

2. and the intervals

$$\hat{F}_i([\bar{T}], [\bar{D}]) = \{\hat{f}_i(t, x) : t \in [\bar{T}], \mathbf{x} \in [\bar{D}] \text{ and}$$

$$\check{F}_i([\bar{T}], [\bar{D}]) = \{\check{f}_i(t, x) : t \in [\bar{T}], \mathbf{x} \in [\bar{D}],$$

$i = 1, 2, \ldots, n$ for each $[\bar{T}] \in [\, \check{t}, \hat{t}\, ]$ and $[\bar{D}] \in [\, \check{\mathbf{d}}, \hat{\mathbf{d}}\, ]$.

At each time step, the solution interval is obtained by determining the smallest interval that will include the largest deviations in the values of $x_i \in [\, \check{d}_i, \hat{d}_i\, ]$. It is should be pointed out that the resulting solution vector will overbound the solutions generated by all possible variations in $[\, \check{t}, \hat{t}\, ]$, $[\, \check{\mathbf{c}}, \hat{\mathbf{c}}\, ]$ and $[\, \check{\mathbf{d}}, \hat{\mathbf{d}}\, ]$.

To prove the convergence of the above algorithm, we proceed as follows:

**Theorem 3.1**: For any non-negative integer $r$,

$$\check{z}_i^{(r+1)} \subset \check{z}_i^{(r)} \quad \text{and} \quad \hat{z}_i^{(r+1)} \subset \hat{z}_i^{(r)}. \tag{3.4}$$

PROOF: See Appendix 3.A.

Next, it will be shown that $\check{\mathbf{s}}$ and $\hat{\mathbf{s}}$ are indeed bounds for the solution set. To show this we will establish that the solutions $x(t)$ are bounded as follows:

**Theorem 3.2**: Given the solution set $\{\mathbf{x}(t)\}$, it is bounded as

$$\check{\mathbf{s}}(t) \;\le\; \{\mathbf{x}(t)\} \;\le\; \hat{\mathbf{s}}(t), \quad t_k \;\in\; [t_k, t_{k+1}]. \tag{3.5}$$

PROOF: See Appendix 3.B.

Knowing the lower and upper bounds of the time response of a linear interval differential equation, feedback can be applied to improve the transient and/or steady

state response of the system. Study of feedback *vis a vis* improvement of time response is of considerable importance and is a possible direction in which to extend the results presented in this report. It is conjectured that similar results can be derived for bounding the gain and phase plots for rational functions with interval coefficients.

## APPENDIX 3.A.

PROOF OF THEOREM 3.1: The proof is by induction. For $r = 0$, we have

$$\check{z}_i^{(1)} = \check{s}_i(t_k) \vee \left(\check{s}_i(t_k) + \check{p}_i^{(0)}h\right) \vee \left(\check{s}_i(t_k) + \check{q}_i^{(0)}h\right), \quad i \;=\; 1, 2, \ldots, n. \tag{3.6}$$

Since, $\check{s}_i(t_k) \in [\check{d}_i, \hat{d}_i]$, $i = 1, 2, \ldots, n$, we can make the increment $h$ sufficiently small such that $\check{z}_i^{(1)} \subset [\check{d}_i, \hat{d}_i] = \check{z}_i^{(0)}$.

Next, assume that $\check{z}_i^{(r)} \subset \check{z}_i^{(r-1)}$ for some $r \geq 2$. Then, in the interval $[t_k, t_{k+1}]$, since $\check{F}_i$ is an inclusion isotone,

$$\begin{aligned}
\left[\check{p}_i^{(r)}, \check{q}_i^{(r)}\right] &= \check{F}_i\left([t_k, t_{k+1}], \check{z}_1^{(r)}, \check{z}_2^{(r)}, \ldots, \check{z}_n^{(r)}\right), \quad i \;=\; 1, 2, \ldots, n. \\
&\subset \check{F}_i\left([t_k, t_{k+1}], \check{z}_1^{(r-1)}, \check{z}_2^{(r-1)}, \ldots, \check{z}_n^{(r-1)}\right) \\
&= \left[\check{p}_i^{(r-1)}, \check{q}_i^{(r-1)}\right]
\end{aligned} \tag{3.7}$$

Therefore,

$$\check{p}_i^{(r-1)} \;\leq\; \check{p}_i^{(r)} \;\leq\; \check{q}_i^{(r)} \;\leq\; \check{q}_i^{(r-1)} \tag{3.8}$$

and,

$$\begin{aligned}
\check{s}_i(t_k) + \check{p}_i^{(r-1)}h \;&\leq\; \check{s}_i(t_k) + \check{p}_i^{(r)}h \\
\check{s}_i(t_k) + \check{q}_i^{(r-1)}h \;&\geq\; \check{s}_i(t_k) + \check{q}_i^{(r)}h.
\end{aligned} \tag{3.9}$$

Equivalently,

$$\check{z}_i^{(r+1)} \subset \check{z}_i^{(r)}. \tag{3.10}$$

In a similar manner, it can be shown that $\hat{z}_i^{(r+1)} \subset \hat{z}_i^{(r)}$. ■

## APPENDIX 3.B.

PROOF OF THEOREM 3.2: For any non-negative integer $r$,

$$\tilde{\check{s}}_i(t) = \check{p}_i^{(r)} \leq \check{f}_i(t, x_1, \ldots, x_n), \quad t \in [t_k, t_{k+1}], \quad x_j \in \check{\mathbf{z}}_j^{(r)}, \quad j = 1, 2, \ldots, n. \tag{3.11}$$

From THEOREM 3.1, it is known that $\check{z}_i^{(r+1)} \subset \check{z}_i^{(r)}$, then for each $t \in [t_k, t_{k+1}]$,

$$\hat{z}_i^{(r+1)} \subset \hat{z}_i^{(r)}. \tag{3.12}$$

$$\tilde{\check{s}}_j(t) = \check{s}_j(t_k) + \check{p}_j^{(r)}(t - t_k) \in \check{z}_j^{(r+1)} \subset \check{z}_j^{(r)}, \quad j = 1, 2, \ldots, n. \tag{3.13}$$

Therefore,

$$\tilde{\check{s}}_i(t) \leq \check{f}_i(t, \check{s}_1(t), \ldots, \check{s}_n(t)), \quad i = 1, 2, \ldots, n. \tag{3.14}$$

Following the proof in an analogous manner, it can be established that

$$\tilde{\hat{s}}_i(t) \geq \hat{f}_i(t, \check{s}_1(t), \ldots, \check{s}_n(t)), \quad i = 1, 2, \ldots, n. \tag{3.15}$$

If we assume that $\mathbf{x}(t)$ is an arbitrary solution of the interval ODE, corresponding to some $\mathbf{c} \in [\mathbf{c}]$ and some $\mathbf{x}_o \in [\mathbf{x}_o]$, then,

$$\begin{aligned} \tilde{\check{\mathbf{s}}}(t) &\leq \check{\mathbf{f}}(t, \check{\mathbf{s}}(t)) \leq \mathbf{f}(t, [\mathbf{c}], \check{\mathbf{s}}(t)) \\ \tilde{\hat{\mathbf{s}}}(t) &\geq \hat{\mathbf{f}}(t, \hat{\mathbf{s}}(t)) \geq \mathbf{f}(t, [\mathbf{c}], \hat{\mathbf{s}}(t)) \\ \check{\mathbf{s}}(t_o) &= \check{\mathbf{x}}_o \leq \mathbf{x}_o \leq \hat{\mathbf{x}}_o = \hat{\mathbf{s}}(t). \end{aligned} \tag{3.16}$$

The above relations can be summarized as

$$\tilde{\check{\mathbf{s}}}(t) \leq \mathbf{f}(t, [\mathbf{c}], \check{\mathbf{s}}(t)),$$

$$\widetilde{\mathbf{x}}(t) = \mathbf{f}(t, [\mathbf{c}], \mathbf{x}(t)),$$

$$\widetilde{\hat{\mathbf{s}}}(t) \geq \mathbf{f}(t, [\mathbf{c}], \hat{\mathbf{s}}(t)),$$

$$\widetilde{\check{\mathbf{s}}}(t_o) \leq \mathbf{x}_o,$$

$$\mathbf{x}(t_o) = \mathbf{x}_o \text{ and}$$

$$\widetilde{\hat{\mathbf{s}}}(t_o) \geq \mathbf{x}_o.$$

Under the assumption that $\mathbf{f}$ is a quasi-isotone in $\mathbf{x}$,

$$\check{\mathbf{s}}(t) \leq \{\mathbf{x}(t)\} \leq \hat{\mathbf{s}}(t). \qquad \blacksquare$$

# 4. A SUFFICIENT CONDITION FOR STABILITY OF A CLASS OF INTERVAL MATRICES

In this Section we derive a sufficient condition for ensuring the stability of a class of interval matrices called $\mathcal{M}$-matrices and study its applications. The Lyapunov criterion of stability of a matrix is extended to interval $\mathcal{M}$-matrix matrix case. However, in extending to the interval case, some conservatism invariably enters the analysis thereby rendering the criterion only sufficient. Some applications of the stability of $\mathcal{M}$-matrices to dynamical systems are presented.

It is well known that for systems described by nominal (intervals with zero width) parameters,

**Theorem 4.1**: Let $\mathbf{V(x)} = \mathbf{x}^T\mathbf{P}\mathbf{x}$, where $\mathbf{P}$ is a solution to the matrix Lyapunov equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}. \tag{4.1}$$

Then a real matrix $\mathbf{A}$ is a stability matrix if and only if, for any given real symmetric positive definite matrix $\mathbf{Q}$, the solution $\mathbf{P}$ is also symmetric positive definite. ■

The solution of a matrix Lyapunov equation can be obtained using existing numerically reliable software for nominal systems. However, these techniques cannot be extended to interval matrices. Fortunately, we can use the following alternate technique for solving the matrix Lyapunov equation.

Define the Kronecker product of two matrices $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{p\times q}$ as $\mathbf{A} \otimes \mathbf{B} \in \mathbb{R}^{mp\times nq}$, where the product is given by

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}. \tag{4.2}$$

Further, we can define

$$\text{vec}\,(\mathbf{P}) = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \vdots \\ \mathbf{p}_n \end{bmatrix} \tag{4.3}$$

where $\mathbf{p}_i$ is the $i$-th column of $\mathbf{P}$. On noting that

$$\text{vec}\,(\mathbf{ABC}) = \mathbf{C}^T \otimes \mathbf{A} \;\; \text{vec}\,(\mathbf{B}) \tag{4.4}$$

and

$$\mathbf{A}^T\mathbf{P} + \mathbf{PA} = \mathbf{A}^T\mathbf{PI} + \mathbf{IPA} = -\mathbf{Q}, \tag{4.5}$$

taking vec $(\cdot)$ on both sides, we get

$$\left(\mathbf{I} \otimes \mathbf{A}^T + \mathbf{A}^T \otimes \mathbf{I}\right) \text{vec}\,(\mathbf{P}) = - \;\text{vec}\,(\mathbf{Q}). \tag{4.6}$$

Equation (4.6) can be solved by Gaussian elimination provided $\left(\mathbf{I} \otimes \mathbf{A}^T + \mathbf{A}^T \otimes \mathbf{I}\right)$ is nonsingular. By the properties of Kronecker sums,† it is clear that the matrix on the left hand side will be nonsingular if and only if $\lambda_i + \lambda_j \neq 0$ for all eigenvalues of $\mathbf{A}$. Clearly, if $\mathbf{A}$ is stable (all eigenvalues in left half plane), then the above condition is satisfied and a unique solution exists.

† The eigenvalues of $\mathbf{A}\otimes\mathbf{B}$ are the $mn$ numbers $\lambda_i\mu_j$, $i = 1, 2, \ldots, n$ $j = 1, 2, \ldots, m$, where $\lambda_i$ and $\mu_j$ are respectively the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$. Further, the eigenvalues of $\mathbf{A} \otimes \mathbf{I}_n + \mathbf{I}_m \otimes \mathbf{B}$ (called the Kronecker sum) are the numbers $\lambda_i + \mu_j$.

For the case of interval systems, we cannot solve for $\mathbf{P}$ directly because it is not easy to compute the inverse of an interval matrix. The best that can be done is obtain an enclosure of the solution vector - vec $(\mathbf{P})$. We next investigate how to compute these enclosures.

## 4.1. Solution of Linear Interval Equations

In this Section we study the problem of solving linear interval equations. These will play a significant role in developing the Lyapunov stability criterion for interval $\mathcal{M}$-matrices. In the literature, a linear interval equation with a coefficient matrix $[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$ and a right hand side $[\,\check{\mathbf{b}},\ \hat{\mathbf{b}}\,]$ is defined as a family of the linear equations characterized by

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{4.7}$$

where $\mathbf{A} \in [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$ and $\mathbf{b} \in [\,\check{\mathbf{b}},\ \hat{\mathbf{b}}\,]$. Note that (4.7) implicitly assumes that $\mathbf{A}$ exists for all $\mathbf{A} \in [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$. The solution set of (4.7) may be defined as

$$\Sigma\Big([\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,],[\,\check{\mathbf{b}},\ \hat{\mathbf{b}}\,]\Big) := \Big\{\mathbf{x} \in \mathbb{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b} \text{ for some } \mathbf{A} \in [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,], \mathbf{b} \in [\,\check{\mathbf{b}},\ \hat{\mathbf{b}}\,]\Big\}. \tag{4.8}$$

An obvious way to compute such a vector $[\,\check{\mathbf{x}},\ \hat{\mathbf{x}}\,]$ is to extend Gaussian elimination to systems of linear equations with interval coefficients. For simplicity of presentation, assume that the (interval) elements of $[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$ are defined as $a_{ij}$ and the elements of $[\,\check{\mathbf{b}},\ \hat{\mathbf{b}}\,]$ are defined as $b_i$. Then starting with the following coefficient tableau:

$$\begin{array}{ccccc} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array} \tag{4.9}$$

and employing the relations:

$$
\begin{aligned}
a'_{1j} &= a_{1j} && 1 \le j \le n \\
a'_{ij} &= a_{ij} - a_{1j}(a_{i1}/a_{11}) && 2 \le i,\ j \le n \\
b'_1 &= b_1 \\
b'_i &= b_i - b_1(a_{i1}/a_{11}) && 2 \le i \le n \\
a'_{i1} &= 0 && 2 \le i \le n.
\end{aligned}
$$

We get the following modified tableau

$$
\begin{array}{ccccc}
a'_{11} & a'_{12} & \cdots & a'_{1n} & b'_1 \\
0 & a'_{22} & \cdots & a'_{2n} & b'_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & a'_{n2} & \cdots & a'_{nn} & b'_n
\end{array}. \tag{4.10}
$$

With the above notation, we can state the following result for general Gaussian elimination:

**Theorem 4.2**: Given $[\ \check{\mathbf{A}},\ \hat{\mathbf{A}}\ ]$ and $[\ \check{\mathbf{b}},\ \hat{\mathbf{b}}\ ]$, the inclusion defined as

$$
\begin{aligned}
\{\mathbf{x}|\mathbf{A}\mathbf{x} &= \mathbf{b},\ \mathbf{A} \in [\ \check{\mathbf{A}},\ \hat{\mathbf{A}}\ ],\ \mathbf{b} \in [\ \check{\mathbf{b}},\ \hat{\mathbf{b}}\ ]\} \\
&\subseteq \{\mathbf{y}|\mathbf{A}'\mathbf{y} = \mathbf{b}',\ \mathbf{A}' \in [\ \check{\mathbf{A}}',\ \hat{\mathbf{A}}'\ ],\ \mathbf{b}' \in [\ \check{\mathbf{b}}',\ \hat{\mathbf{b}}'\ ]\}
\end{aligned} \tag{4.11}
$$

is valid.

PROOF: See Appendix 4.A.

Now using the above result, if we carry out the Gaussian elimination for the remaining columns, then the coefficient tableau is transformed to an upper triangular form:

$$
\begin{array}{ccccc}
\tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} & \tilde{b}_1 \\
0 & \tilde{a}_{22} & \cdots & \tilde{a}_{2n} & \tilde{b}_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \tilde{a}_{nn} & \tilde{b}_n
\end{array}. \tag{4.12}
$$

for which

$$\{\mathbf{x}|\mathbf{Ax} = \mathbf{b},\ \mathbf{A} \in [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,],\ \mathbf{b} \in [\,\check{\mathbf{b}}.\ \hat{\mathbf{b}}\,]\}$$
$$\subseteq \{\tilde{\mathbf{y}}|\tilde{\mathbf{A}}'\tilde{\mathbf{y}} = \tilde{\mathbf{b}}',\ \tilde{\mathbf{A}}' \in [\,\widetilde{\check{\mathbf{A}}',\ \hat{\mathbf{A}}'}\,],\ \tilde{\mathbf{b}}' \in [\,\widetilde{\check{\mathbf{b}}',\ \hat{\mathbf{b}}'}\,]\} \tag{4.13}$$

is valid. Using the relations

$$\begin{aligned} x_n &= \tilde{b}_n/\tilde{a}_{nn} \\ x_i &= \frac{\tilde{b}_i - \sum_{j=i+1}^{n} \tilde{a}_{ij}x_j}{\tilde{a}_{ii}}, \quad 1 \le i \le n-1, \end{aligned} \tag{4.14}$$

it is possible to obtain a vector $[\,\check{\mathbf{x}},\ \hat{\mathbf{x}}\,] = (x_i)$ such that

$$\{\mathbf{x}|\mathbf{Ax} = \mathbf{b},\ \mathbf{A} \in [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,],\ \mathbf{b} \in [\,\check{\mathbf{b}},\ \hat{\mathbf{b}}\,]\} \subseteq [\,\check{\mathbf{x}},\ \hat{\mathbf{x}}\,]\} \tag{4.15}$$

**Theorem 4.3:** Let $1 \le n \le 2$ and assume that the $n \times n$ interval matrix $[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,] = ([\,\check{a}_{ij},\ \hat{a}_{ij}\,])$ does not contain a singular matrix $\mathbf{A}$, then the Gaussian algorithm can always be carried out.

PROOF: See Appendix 4.B.

Unfortunately, the above result is true for $n \times n$ interval matrices where $1 \le n \le 2$ only and cannot be extended to the case when $n \ge 3$. Note that Gaussian elimination *can* be performed for the several special classes of matrices. One of the more important ones being $\mathcal{M}$-matrices. In the next section we will show that if the interval matrix is also an $\mathcal{M}$-matrix, then the solution of linear interval equations is considerably simplified.

## 4.2. Convex Hull of $\mathcal{M}$-Matrices

We will consider the case of square interval matrices. The results can also be extended to non-square interval matrices, with applications in solving interval least square problems.

**Definition 4.4**: A square matrix $\mathbf{A}$ is called an $\mathcal{M}$-matrix if $a_{ij} \le 0$ for $i \ne j$ and $\mathbf{Au} > 0$ for some positive vector $\mathbf{u} \in \mathbb{R}^n$.

An important property of nonsingular $\mathcal{M}$-matrices is that every element of their inverse is non-negative.

As mentioned earlier, the computation of $[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]^{-1}$ for a regular† interval matrix is a difficult problem. The known methods have a computational complexity exponential in $n$ (the order of the matrix). But for the important special case of $\mathcal{M}$-matrices, we have the following explicit result [28], [29]:

**Theorem 4.5**: For an $n \times n$ interval matrix $[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$, if the matrices $\check{\mathbf{A}}$ and $\hat{\mathbf{A}}$ are regular and $\check{\mathbf{A}}^{-1} \ge 0$, $\hat{\mathbf{A}}^{-1} \ge 0$, then $[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$ is regular and

$$[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]^{-1} = \left[\hat{\mathbf{A}}^{-1},\ \check{\mathbf{A}}^{-1}\right]. \tag{4.16}$$

Proof: See Appendix 4.C.

## 4.3. Stability of Interval $\mathcal{M}$-Matrices

Researchers have attempted to derive necessary and sufficient conditions for establishing the Hurwitz or Schur stability of interval matrices. However, invariably, the results have been shown true for only a smaller class of matrices or the conditions

† If all matrices $\mathbf{A} \in [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$ have full rank, then $[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$ is a regular matrix

have been proved only to be sufficient. Notably, Białas [10] presented a result that derived a necessary and sufficient condition of stability of general interval matrices. This result was later disproved by several researchers *e.g.*, Barmish and Hollot [6] and Karl *et al.*, [20]. A sufficient condition for stability of interval matrices is derived by Heinen [17] and Yedavalli [43]. In [15], the authors obtain the conditions for stability of a special class of matrices. Other related work has been in obtaining the perturbation bounds on the elements of a nominal stable matrix. These bounds can be found in the work of Patel, Toda and Sridhar [31], [32] and Yedavalli [42].

In this Section, we will derive the condition for the stability of interval $\mathcal{M}$-matrices. The following important properties of the $\mathcal{M}$-matrices will be used in the sequel:

**Theorem 4.6**: POSITIVE STABILITY [8] $\mathbf{A}$ is a nonsingular $\mathcal{M}$-matrix if and only if there exists a solution $\mathbf{W}$ to the matrix equation

$$\mathbf{AW} + \mathbf{WA}^T = \mathbf{Q} \tag{4.17}$$

where $\mathbf{W}$ and $\mathbf{Q}$ are symmetric positive definite matrices. ■

**Theorem 4.7**: POSITIVITY OF PRINCIPAL MINORS [8] $\mathbf{A}$ is a nonsingular $\mathcal{M}$-matrix if and only if $\mathbf{A}$ is nonsingular and $\mathbf{x} \neq \mathbf{0}$; $\mathbf{y} = \mathbf{Ax}$, then for some subscript $i$, $x_i \neq 0$, and $x_i y_i \geq 0$. ■

**Theorem 4.8**: If $\mathbf{A}$ is a nonsingular $\mathcal{M}$-matrix then $(\mathbf{I} \otimes \mathbf{A}^T + \mathbf{A}^T \otimes \mathbf{I})$ is also a nonsingular $\mathcal{M}$-matrix.

PROOF: See Appendix 4.D.

The above result is fairly important for the sequel, hence it is illustrated by means of a simple numerical example.

**Example 4.1**: Consider the $(2 \times 2)$ non-singular $\mathcal{M}$-matrix

$$\mathbf{A} = \begin{bmatrix} 5 & -1 \\ -2 & 4 \end{bmatrix}.$$

Forming the Kroneckar products, we get

$$\mathbf{I} \otimes \mathbf{A}^T = \begin{bmatrix} 5 & -2 & 0 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -1 & 4 \end{bmatrix}, \quad \mathbf{A}^T \otimes \mathbf{I} = \begin{bmatrix} 5 & 0 & -2 & 0 \\ 0 & 5 & 0 & -2 \\ -1 & 0 & 4 & 0 \\ 0 & -1 & 0 & 4 \end{bmatrix}.$$

Therefore,

$$\mathbf{I} \otimes \mathbf{A}^T + \mathbf{A}^T \otimes \mathbf{I} = \begin{bmatrix} 10 & -2 & -2 & 0 \\ -1 & 9 & 0 & -2 \\ -1 & 0 & 9 & -2 \\ 0 & -1 & -1 & 8 \end{bmatrix}.$$

The inverse of the above matrix is

$$\begin{bmatrix} 1.0494D-01 & 2.4691D-02 & 2.4691D-02 & 1.2346D-02 \\ 1.2346D-02 & 1.1728D-01 & 6.1728D-03 & 3.0864D-02 \\ 1.2346D-02 & 6.1728D-03 & 1.1728D-01 & 3.0864D-02 \\ 3.0864D-03 & 1.5432D-02 & 1.5432D-02 & 1.3272D-01 \end{bmatrix}.$$

Thereby, verifying that $\mathbf{I} \otimes \mathbf{A}^T + \mathbf{A}^T \otimes \mathbf{I}$ is a non-singular $\mathcal{M}$-matrix.

Next, we determine the condition for the ***positive stability*** of interval $\mathcal{M}$-matrices. By definition, all eigenvalues of $[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$ are positive (*i.e.* the matrix is positive stable). Therefore, to obtain a Lyapunov function to establish positive stability of interval matrices, we need to find an interval positive definite matrix as a solution of the following Lyapunov equation

$$[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]^T[\,\check{\mathbf{P}},\ \hat{\mathbf{P}}\,] + [\,\check{\mathbf{P}},\ \hat{\mathbf{P}}\,][\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,] = \mathbf{Q}. \tag{4.18}$$

where $\mathbf{Q}$ is positive definite matrix. Then, if each $\mathbf{P} \in [\,\check{\mathbf{P}},\ \hat{\mathbf{P}}\,]$ is a positive definite matrix, $[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$ is a positive stable matrix.

Equivalently, assuming that $[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$ is an interval $\mathcal{M}$-matrix, we form the Kronecker sum given by (4.2) for interval matrix case and solve the following linear interval equation for vec $(\mathbf{P})$:

$$\left(\mathbf{I} \otimes [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]^T + [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]^T \otimes \mathbf{I}\right) \ \mathrm{vec}\,([\,\check{\mathbf{P}},\ \hat{\mathbf{P}}\,]) = \mathrm{vec}\,(\mathbf{Q}). \tag{4.19}$$

Denoting the $n^2 \times n^2$ matrix on the left hand side by $[\,\check{\mathcal{A}},\ \hat{\mathcal{A}}\,]$, we have

$$[\,\check{\mathcal{A}},\ \hat{\mathcal{A}}\,] \ \mathrm{vec}\,([\,\check{\mathbf{P}},\ \hat{\mathbf{P}}\,]) = \mathrm{vec}\,(\mathbf{Q}). \tag{4.20}$$

where vec $(\mathbf{Q}) > 0$. Now, using the properties of interval $\mathcal{M}$-matrices,

$$\begin{aligned} \mathrm{vec}\,([\,\check{\mathbf{P}},\ \hat{\mathbf{P}}\,]) &= [\,\check{\mathcal{A}},\ \hat{\mathcal{A}}\,]^{-1} \mathrm{vec}\,(\mathbf{Q}) \\ &= [\,\hat{\mathcal{A}}^{-1},\ \check{\mathcal{A}}^{-1}\,] \mathrm{vec}\,(\mathbf{Q}) \\ &= [\,\hat{\mathcal{A}}^{-1} \mathrm{vec}\,(\mathbf{Q}),\ \check{\mathcal{A}}^{-1} \mathrm{vec}\,(\mathbf{Q})\,]. \end{aligned} \tag{4.21}$$

The interval matrix $[\,\check{\mathbf{P}},\ \hat{\mathbf{P}}\,]$ can now be easily constructed. Clearly, $[\,\check{\mathbf{P}},\ \hat{\mathbf{P}}\,]$ is a symmetric matrix.

To verify whether it is positive definite, we use the result developed by Shi and Gao [38]. Following the notation used in [38], let $\mathbf{U} = (u_{ij})$ and $\mathbf{V} = (v_{ij}) \in \mathbb{R}^{n \times m}$ be symmetric matrices and $u_{ij} \le v_{ij},\ i, j = 1, 2, \ldots, n$. Denoting

$$\begin{aligned} \mathcal{L}^n[\mathbf{U}, \mathbf{V}] &= \{\mathbf{A} = (a_{ij}) \in \mathbb{R}^{n \times n} | u_{ij} \le a_{ij} \le v_{ij};\ a_{ij} = a_{ji}, i, j = 1, 2, \ldots, n\} \\ \mathcal{K}^n[\mathbf{U}, \mathbf{V}] &= \{\mathbf{S} = (s_{ij}) \in \mathbb{R}^{n \times n} | s_{ij} = u_{ij} \text{ or } s_{ij} = v_{ij};\ s_{ij} = s_{ji}, i, j = 1, 2, \ldots, n\}. \end{aligned} \tag{4.22}$$

Note that while the first equation represents a set of matrices with infinite members, the second equation denotes a set of edge matrices. According to [38], an interval matrix $\mathcal{L}^n[\mathbf{U},\mathbf{V}]$ ($\mathcal{K}^n[\mathbf{U},\mathbf{V}]$) is termed a positive definite matrix if $\mathbf{A}$ ($\mathbf{S}$) is positive definite for every $\mathbf{A} \in \mathcal{L}^n[\mathbf{U},\mathbf{V}]$ ($\mathbf{S} \in \mathcal{K}^n[\mathbf{U},\mathbf{V}]$). Based on the above notation, the following result was established in [38]:

**Theorem 4.9**: The set $\mathcal{L}^n[\mathbf{U},\mathbf{V}]$ consists of positive definite matrices if and only if the set $\mathcal{K}^n[\mathbf{U},\mathbf{V}]$) consists of only positive definite matrices. ■

Using the above result, now it is clear that [ $\check{\mathbf{P}}$, $\hat{\mathbf{P}}$ ], the solution of interval Lyapunov equation is positive definite if and only if all the vertex matrices are positive definite. To establish the positive definiteness one would have to verify positive definiteness of all vertex matrices. Note, however, the matrix $\mathcal{A}$ is (4.20) is over-bounded, hence positive definiteness of [ $\check{\mathbf{P}}$, $\hat{\mathbf{P}}$ ] is only sufficient for *positive stability* of [ $\check{\mathbf{A}}$, $\hat{\mathbf{A}}$ ]. Note further that if the entire interval Lyapunov equation is premultiplied with negative identity, parallel results can be obtained for *Hurwitz stability* of inverse negative matrices (matrices whose inverse has all negative elements).

## 4.4. Applications of Stability of $\mathcal{M}$-Matrices

In this Section, we study two applications of the stability conditions developed above. The first application is in the study of robust stability of composite dynamical systems and the second application is in the study of robustness of dynamical Leontief economic models.

### 4.4.1. ROBUST STABILITY OF DECENTRALIZED SYSTEMS

The earliest application of $\mathcal{M}$-matrix theory to composite dynamical system stability

problem was studied by Gurjić and Šiljak [18]; and Araki [2]. In the sequel we will follow the notation used by Gurjić and Šiljak. Consider a continuous time system described by a set of $n$ vector differential equations [37]:

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{g}_i(\mathbf{x}_i, t) + \mathbf{h}(\mathcal{X}, t), \quad i = 1, 2, \ldots, k \tag{4.23}$$

where $\mathbf{x}_i \in \mathbb{R}^{n_i}$, $\mathbf{g}_i : \mathbb{R}^{n_i} \times \mathbb{R} \to \mathbb{R}^{n_i}$, $\mathbf{h}_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_k} \times \mathbb{R} \to \mathbb{R}^{n_i}$ and $\mathcal{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$.

Assume that

$$\mathbf{g}_i(\mathbf{0}, t) = \mathbf{0} \quad \forall\ t \in \mathbb{R} \tag{4.24a}$$

$$\mathbf{h}_i(\mathbf{0}, \ldots, \mathbf{0}, t) = \mathbf{0} \quad \forall\ t \in \mathbb{R} \tag{4.24b}$$

*i.e.*, the null state is the equilibrium state. Clearly, the above system can be regarded as a composite system consisting of $k$ subsystems give by

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{g}_i(\mathbf{x}_i, t) + \mathbf{y}_i, \quad i = 1, 2, \ldots, k \tag{4.25}$$

where $\mathbf{y}_i$ denotes the interconnecting relations between the various stations. When the interaction $\mathbf{h}_i(\mathcal{X}, t) = \mathbf{0}$, the unforced subsystem is defined as

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{g}_i(\mathbf{x}_i, t). \tag{4.26}$$

In the subsequent paragraphs we will assume that the stability of each unforced system has been established by determining scalar Lyapunov functions $v_i : \mathbb{R} \times \mathbb{R}^{n_i} \to \mathbb{R}$ satisfying

$$\phi_{i1}(\|\mathbf{x}_i\|) \leq v_i(t, \mathbf{x}_i) \leq \phi_{i2}(\|\mathbf{x}_i\|) \quad \text{and} \tag{4.27a}$$

$$\frac{dv_i(t, \mathbf{x}_i)}{dt} \leq \phi_{i3}(\|\mathbf{x}_i\|) \quad \forall\ t \in \mathbb{R} \quad \text{and} \quad \forall\ \mathbf{x}_i \in \mathbb{R}^{n_i}. \tag{4.27b}$$

The total time derivative of $v_i(\mathbf{x}_i, t)$ is given by

$$v_i = \frac{\partial}{\partial t} v_i + (\text{grad } v_i)^T \mathbf{g}_i. \tag{4.28}$$

Further, it is assumed that there exist bounded functions $\xi ij : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$,

$$\sup_{\mathbb{R}\times\mathbb{R}^n} |\xi_{ij}| = v_{ij} \leq \infty \quad \forall \; i, j = 1, 2, \ldots, k \tag{4.29}$$

subject to the following inequalities:

$$(\text{grad } v_i)^T \mathbf{h}_i \leq \sum_{j=1}^{k} \xi_{ij}(t, \mathcal{X} \phi_{i3}(\|\mathbf{x}_i\|) \quad i = 1, 2, \ldots, k. \tag{4.30}$$

Next, we study the composite stability of the system, knowing the stability properties of individual subsystems. The total time derivative of the function $v_i(\mathcal{X}, t)$ along the solutions of unforced individual subsystems is given by

$$\dot{\tilde{v}}_i = \dot{v}_i + (\text{grad } v_i)^T \mathbf{h}_i. \tag{4.31}$$

Using the bounds in (4.27) and (4.30), we get

$$\dot{\tilde{v}}_i \leq \phi_{i3}(\|\mathbf{x}_i\|) + \sum_{j=1}^{k} \xi_{ij}(t, \mathcal{X} \phi_{i3}(\|\mathbf{x}_i\|) \quad i = 1, 2, \ldots, k. \tag{4.32}$$

Denoting the vector Lyapunov function as $\mathbf{v} = [v_1, \; \cdots \; , v_k]^T$ and the comparison vector function $\Phi = [\phi_{13}, \phi_{12}, \; \cdots \; , \phi_{k3}]^T$ related as

$$\dot{\mathbf{v}} \leq \mathbf{A}\Phi, \tag{4.33}$$

the elements of $\mathbf{A}$ are given by (4.32) as

$$a_{ij} = -\delta_{ij} + \alpha_{ij} \tag{4.34}$$

where, $\delta_{ij}$ is the Kronecker symbol and

$$\mathbf{a}_{ij} = \delta_{ij} \sup_{\mathbb{R}^n \times \mathbb{R}} \xi_{ij}(\mathcal{X}, t) + \max\{0, \sup_{\mathbb{R}^n \times \mathbb{R}} \xi_{ij}(\mathcal{X}, t)\}. \tag{4.35}$$

Clearly, the stability of the composite system is governed by the stability of the dynamical system defined by (4.33). Notice that in (4.33), $\mathbf{A}$ is an $\mathcal{M}$-matrix.

Now, to illustrate the use of the theory developed in the previous Section, let us consider the example in [18].

**Example 4.2**: Assume that the system is composed of two subsystems defined by

$$\frac{d\mathbf{x}_1}{dt} = \mathbf{g}_1(\mathbf{x}_1) + \mathbf{h}_1(\mathcal{X}) \tag{4.36a}$$

$$\frac{d\mathbf{x}_2}{dt} = \mathbf{g}_1(\mathbf{x}_1, t) + \mathbf{h}_1(\mathcal{X}, t) \tag{4.36b}$$

where $\mathcal{X} = [\mathbf{x}_1^T \ \mathbf{x}_2^T]^T$, $\mathbf{x}_1 = [x_{11} \ x_{12}]^T$, $\mathbf{x}_2 = [x_{21} \ x_{22}]^T$,

$$\mathbf{g}_1 = \begin{bmatrix} x_{12} \\ -2x_{11} - x_{11}^2 x_{12} \end{bmatrix} \tag{4.37a}$$

$$\mathbf{h}_1 = \begin{bmatrix} \frac{1}{2}[\,\check{\alpha},\ \hat{\alpha}\,]x_{11}x_{12}^2 + [\,\check{\beta},\ \hat{\beta}\,](x_{21}^3 + x_{22}^3) \\ [\,\check{\alpha},\ \hat{\alpha}\,]x_{11}^2 x_{12} + [\,\check{\beta},\ \hat{\beta}\,](x_{21}^3 + x_{22}^3) \end{bmatrix} \tag{4.37b}$$

$$\mathbf{g}_2 = \begin{bmatrix} x_{22}\cos(t) - x_{21}(x_{21}^2 + x_{22}^2)(2 + \sin(t)) \\ -x_{21}\cos(t) - x_{22}(x_{21}^2 + x_{22}^2)(2 + \sin(t)) \end{bmatrix} \tag{4.37c}$$

$$\mathbf{h}_2 = \begin{bmatrix} [\,\check{\delta},\ \hat{\delta}\,]x_{21}(x_{21}^2 + x_{22}^2) + \theta(\mathbf{x}_1, t) \\ [\,\check{\delta},\ \hat{\delta}\,]x_{22}(x_{21}^2 + x_{22}^2) + \theta(\mathbf{x}_1, t) \end{bmatrix} \tag{4.37d}$$

where $|\theta(\mathbf{x}_1, t)| \le [\,\check{\gamma},\ \hat{\gamma}\,]\phi_{i3}(\|\mathbf{x}_1\|)$ and $[\,\check{\alpha},\ \hat{\alpha}\,]$, $[\,\check{\beta},\ \hat{\beta}\,]$, $[\,\check{\gamma},\ \hat{\gamma}\,]$ and $[\,\check{\delta},\ \hat{\delta}\,]$ are all positive interval parameters. The interval parameters are a consequence of uncertainties in the interconnection models. Selecting

$$v_1(\mathbf{x}_1) = [0.5(2x_{11}^2 + x_{12}^2)]^{1/2}, \quad v_2(\mathbf{x}_2) = \|\mathbf{x}_2\| \tag{4.38}$$

and following equations (4.30)-(4.35), we get

$$\dot{\tilde{v}}_1 \leq -(1 - 8[\check{\alpha}, \hat{\alpha}])\phi_{13}(\|\mathbf{x}_1\|) + 8[\check{\beta}, \hat{\beta}]\phi_{23}(\|\mathbf{x}_2\|) \tag{4.39a}$$

$$\dot{\tilde{v}}_2 \leq 2[\check{\gamma}, \hat{\gamma}])\phi_{13}(\|\mathbf{x}_1\|) - (1 - [\check{\delta}, \hat{\delta}])\phi_{23}(\|\mathbf{x}_2\|). \tag{4.39b}$$

Equivalently,

$$\mathbf{v} \leq \begin{bmatrix} -1 + 8[\check{\alpha}, \hat{\alpha}] & 8[\check{\beta}, \hat{\beta}] \\ 2[\check{\gamma}, \hat{\gamma}] & -1 + [\check{\delta}, \hat{\delta}] \end{bmatrix} \Phi \tag{4.40}$$

Notice that, since $[\check{\mathbf{A}}, \hat{\mathbf{A}}]$ in (4.40) is only a $2 \times 2$ matrix, the stability is guaranteed provided both diagonal elements are negative. However, in general it will be difficult to ensure the stability conditions.

Fortunately, $[\check{\mathbf{A}}, \hat{\mathbf{A}}]$ is an interval $\mathcal{M}$-matrix, thereby, simplifying the analysis. To apply interval matrix analysis developed in previous Section, we study the *positive stability* of $-[\check{\mathbf{A}}, \hat{\mathbf{A}}]$. If $-[\check{\mathbf{A}}, \hat{\mathbf{A}}]$ is positive stable, then $[\check{\mathbf{A}}, \hat{\mathbf{A}}]$ will have *Hurwitz stability*.

### 4.4.2. DYNAMIC LEONTIEF MODELS

One of the simpler static economical models [41] has the following characteristics:

1. no joint production; each sector or industry produces one and only one commodity, *i.e.*, there is one to one correspondence between the sectors and the commodities
2. each sector has only one technique of production; to produce one unit of commodity $j$, sector $j$ requires $a_{ij}$ units of commodity $i$; $i, j = 1, 2, \ldots, n$
3. there are no production lags
4. there is no government activity or foreign trade

5. the model allows for only those commodities which cease to exist in the industry, once they are used in the production process.

With these assumptions on the activity, and denoting the following for $i, j = 1, 2, \ldots, n$:

$x_i$ = gross output of sector $i$

$x_{ij}$ = amount of commodity $i$ used by the sector $j$

$a_{ij} = x_{ij}/x_j$

$c_i$ = final demand for commodity $i$

the balance equations for each sector can be written as

$$x_i = \sum_{j=1}^{n} x_{ij} + c_i, \quad i = 1, 2, \ldots, n \tag{4.41a}$$

$$x_i = \sum_{j=1}^{n} a_{ij} x_j + c_i, \tag{4.41b}$$

$$\Leftrightarrow \quad \mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{c}, \tag{4.41c}$$

where $a_{ij}$ is the $(i, j)$-th element of the matrix $\mathbf{A}$. The model in (4.41) is the static economic model. The properties of the static model can be studied from the above system of linear equations.

This model can be further generalized to the dynamic model. For the dynamic model, we modify assumption (2) as follows — to produce one unit of commodity $j$, sector $j$ requires $a_{ij}$ units as the current input of commodity $i$ and $b_{ij}$ units as the capital input. Then defining

$s_{ij}$ = the stock of commodity $i$ held by sector $j$

$b_{ij} = s_{ij}/x_j$,

the continuous time balance equations for each sector can be rewritten as

$$x_i = \sum_{j=1}^{n} x_{ij} + \sum_{j=1}^{n} \frac{ds_{ij}}{dt} + c_i, \quad i = 1, 2, \ldots, n \tag{4.42a}$$

$$x_i = \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} b_{ij}\frac{dx_j}{dt} + c_i, \quad i = 1, 2, \ldots, n \tag{4.42a}$$

$$\Leftrightarrow \quad \mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\frac{d\mathbf{x}}{dt} + \mathbf{c}, \tag{4.42c}$$

The above model can be expressed into the standard state space for as

$$\frac{d\mathbf{x}}{dt} = \mathbf{B}^{-1}(\mathbf{I} - \mathbf{A})\mathbf{x} + \mathbf{B}^{-1}\mathbf{c} \tag{4.43}$$

In general, the capital input is known very accurately. Allowing perturbations in the input of the current commodity $a_{ij}$, we have the following interval plant:

$$\frac{d\mathbf{x}}{dt} = \mathbf{B}^{-1}(\mathbf{I} - [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,])\mathbf{x} + \mathbf{B}^{-1}\mathbf{c} \tag{4.44}$$

where $\mathbf{I} - [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$ is a non-singular interval $\mathcal{M}$-matrix, therefore we can apply the results of Sections 4.1-4.3 and study the stability of dynamic Leontief models.

Next, we present an example to illustrate the robust stability of uncertain Leontief models.

**Example 4.3**: Following the above notation, assume that the current commodity has some uncertainty and the capital input is known precisely. The corresponding matrices $[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$ and $\mathbf{B}$ are given by:

$$[\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,] = \begin{bmatrix} [0.05, 0.10] & [0.10, 0.30] & [0.20, 0.25] \\ [0.00, 0.10] & [0.15, 0.20] & [0.20, 0.30] \\ [0.10, 0.20] & [0.10, 0.15] & [0.05, 0.10] \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then the system matrix is given by

$$\mathbf{B}^{-1}\left(\mathbf{I} - [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]\right) = \begin{bmatrix} [+0.90, +0.95] & [-0.30, -0.10] & [-0.25, -0.20] \\ [-0.10, +0.00] & [+0.80, +0.85] & [-0.30, -0.20] \\ [-0.20, -0.10] & [-0.15, -0.10] & [+0.90, +0.95] \end{bmatrix}.$$

Solving the associated interval Lyapunov equations with $\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we have

$$[\,\check{\mathbf{P}},\ \hat{\mathbf{P}}\,] = \begin{bmatrix} [0.5521,\ 0.6988] & [0.0553,\ 0.2432] & [0.0946,\ 0.2238] \\ [0.0553,\ 0.2432] & [0.6112,\ 0.7380] & [0.0978,\ 0.2202] \\ [0.0946,\ 0.2238] & [0.0978,\ 0.2202] & [0.5466,\ 0.6420] \end{bmatrix}$$

which is a symmetric matrix (rounded to 4 decimal places). Notice that it is diagonally dominant; therefore, one can expect favorable results. However, using the result in [38], it was verified that the vertex matrices of $[\,\check{\mathbf{P}},\ \hat{\mathbf{P}}\,]$ are indeed positive definite. Therefore the system exhibits positive stability. Since $[\,\check{\mathbf{P}},\ \acute{\mathbf{P}}\,]$ is a symmetric matrix of order 3, there will be $2^6$ edge matrices. In order to save space, we have given below the plots of the eigenvalues of $[\,\check{\mathbf{P}},\ \hat{\mathbf{P}}\,]$. FIGURE 4.1 plots the three eigenvalues as the elements of the matrix $[\,\check{\mathbf{P}},\ \hat{\mathbf{P}}\,]$ are varied. The matrices are in the following sequence: the first matrix $\mathcal{P}_1$ is the matrix of lower bounds of each interval *i.e.* $\cap P_1 = \check{\mathbf{P}}$. In the second matrix, the $(1,1)$ element is the upper bound of $[\,\check{p}_{11},\ \hat{p}_{11}\,]$ and the rest of the elements are the lower bounds,

$$\mathcal{P}_2 = \begin{bmatrix} 0.6988 & 0.0553 & 0.0946 \\ 0.0553 & 0.6112 & 0.0978 \\ 0.0946 & 0.0978 & 0.5466 \end{bmatrix}.$$

The third matrix has its $(1,2)$ and $(2,1)$ elements as the upper bound of $[\,\check{p}_{12},\ \hat{p}_{12}\,]$ and the rest of the elements are the lower bounds.

$$\mathcal{P}_3 = \begin{bmatrix} 0.5521 & 0.2432 & 0.0946 \\ 0.2432 & 0.6112 & 0.0978 \\ 0.0946 & 0.0978 & 0.5466 \end{bmatrix}.$$

In this manner elements are varied till the matrix becomes $\hat{\mathbf{P}}$. All eigenvalues of each matrix are positive, thereby establishing positive definiteness of the symmetric interval matrix $[\,\check{\mathbf{P}},\ \hat{\mathbf{P}}\,]$.

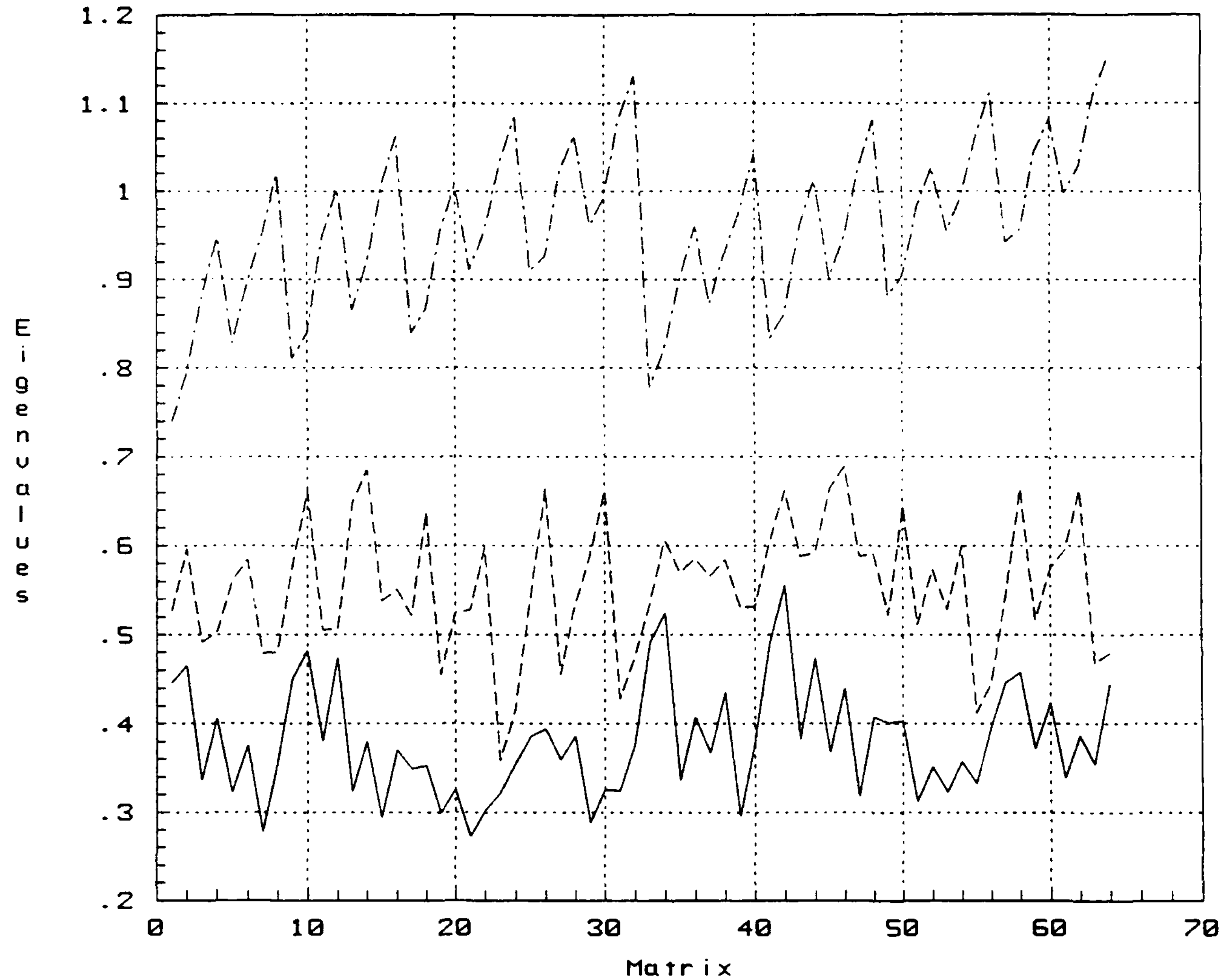


**Figure 4.1:** EIGENVALUES OF VARIOUS EDGE MATRICES

## APPENDIX 4.A.

PROOF OF THEOREM 4.2: Assume that $\mathbf{A} \in [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$ and $\mathbf{b} \in [\,\check{\mathbf{b}},\ \hat{\mathbf{b}}\,]$ and consider the system of linear equations

$$\mathbf{Ax} = \mathbf{b}. \tag{4.45}$$

Let $\mathbf{A} = (a_{ij})$, the vector $\mathbf{b} = (b_i)$, $\mathbf{A}' = (a'_{ij})$ and the vector $\mathbf{b}' = (b'_i)$, where,

$$\begin{aligned}
a'_{1j} &= a_{1j} && 1 \le j \le n \\
a'_{ij} &= a_{ij} - a_{1j}(a_{i1}/a_{11}) && 2 \le i,\ j \le n \\
b'_1 &= b_1 && \\
b'_i &= b_i - b_1(a_{i1}/a_{11}) && 2 \le i \le n \\
a'_{i1} &= 0 && 2 \le i \le n
\end{aligned}$$

where $0 \notin a_{11}$. Clearly $\mathbf{A}'\mathbf{y} = \mathbf{b}'$ has the same solution as $\mathbf{Ax} = \mathbf{b}$. Then by the inclusion monotonicity (DEFINITION 2.5), of interval operations, it follows that

$$\mathbf{A}' \in [\,\check{\mathbf{A}}',\ \hat{\mathbf{A}}'\,] \quad \text{and} \quad \mathbf{b}' \in [\,\check{\mathbf{b}}',\ \hat{\mathbf{b}}'\,]. \tag{4.46}$$

which proves the assertion. ■

## APPENDIX 4.B.

PROOF OF THEOREM 4.3: For $n = 1$, the assumption that $0 \notin [\,\check{a}_{11},\ \hat{a}_{11}\,]$ directly proves the assertion. For $n = 2$, at least one of the intervals $[\,\check{a}_{11},\ \hat{a}_{11}\,]$ and $[\,\check{a}_{21},\ \hat{a}_{21}\,]$ must not contain zero. If this was not the case then there would exist a singular matrix

$\mathbf{A} \in [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$, contradicting the assumption of the theorem. Assume without loss of generality, $0 \notin [\,\check{a}_{11},\ \hat{a}_{11}\,]$. Using the Gaussian elimination, we have

$$a'_{22} = a_{22} - (1/a_{11})a_{21}a_{12}. \tag{4.47}$$

Let $a_{ij} \in [\,\check{a}_{ij},\ \hat{a}_{ij}\,]$. Then

$$a'_{22} = a_{22} - (1/a_{11})a_{21}a_{12}. \tag{4.48}$$

But, by assumption,

$$\det(\mathbf{A}) = a_{11}a_{22} - a_{21}a_{12} \neq 0. \tag{4.49}$$

Therefore,

$$a'_{22} = (1/a_{11})\det(\mathbf{A}) \neq 0.$$ ∎

## APPENDIX 4.C.

PROOF OF THEOREM 4.5: Select any arbitrary vector $\mathbf{v} > 0$. Then, since $\check{\mathbf{A}}^{-1} > 0$ and it is regular, $\mathbf{u} := \check{\mathbf{A}}^{-1}\mathbf{v} > 0$. Next, let $\mathbf{A} \in [\,\check{\mathbf{A}},\ \hat{\mathbf{A}}\,]$, obviously $\check{\mathbf{A}} < \mathbf{A} < \hat{\mathbf{A}}$. Then,

$$\hat{\mathbf{A}}^{-1}\mathbf{A} \leq \mathbf{I} \leq \check{\mathbf{A}}^{-1}\mathbf{A}. \tag{4.50}$$

Therefore, $\mathbf{B} := \hat{\mathbf{A}}^{-1}\mathbf{A}$ satisfies $\mathbf{B} \leq \mathbf{I}$ and $\mathbf{B}\mathbf{u} = \hat{\mathbf{A}}^{-1}\mathbf{A}\mathbf{u} \geq \hat{\mathbf{A}}^{-1}\check{\mathbf{A}}\mathbf{u} = \hat{\mathbf{A}}^{-1}\mathbf{v} > 0$. This implies that $\mathbf{B}$ is an $\mathcal{M}$-matrix and both $\mathbf{B}$ and $\mathbf{A} = \hat{\mathbf{A}}\mathbf{B}$ are regular.

Now, since $\mathbf{A}^{-1} = \mathbf{B}^{-1}\hat{\mathbf{A}}^{-1} \geq 0$,

$$\hat{\mathbf{A}}^{-1} \leq \mathbf{A}^{-1} \leq \check{\mathbf{A}}^{-1}, \tag{4.51}$$

where the equality holds for $\mathbf{A} = \hat{\mathbf{A}}$ and $\mathbf{A} = \check{\mathbf{A}}$. ∎

## APPENDIX 4.D.

PROOF OF THEOREM 4.8: If $\mathbf{A}$ is a nonsingular $\mathcal{M}$-matrix then there exists a positive definite $\mathbf{W}$ such that

$$\mathbf{AW} + \mathbf{WA}^T$$

is positive definite. Since a solution $\mathbf{W}$ exists, then it is clear that $(\mathbf{I}\otimes\mathbf{A}^T + \mathbf{A}^T\otimes\mathbf{I})$ is a nonsingular matrix. To prove the result, it suffices to show that if $\mathbf{x} \neq \mathbf{0}$; $\mathbf{y} = \mathbf{Ax}$, then for some subscript $i$, $x_i \neq 0$, and $x_i y_i \geq 0$.

For simplicity of notation, let $\mathbf{B} = \mathbf{A}^T$. The explicit form of $\mathcal{B} = (\mathbf{I}\otimes\mathbf{B} + \mathbf{B}\otimes\mathbf{I})$ is given by

$$\begin{aligned}\mathcal{B} &= \begin{bmatrix} \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B} \end{bmatrix} + \begin{bmatrix} b_{11}\mathbf{I} & b_{12}\mathbf{I} & \cdots & b_{1n}\mathbf{I} \\ b_{21}\mathbf{I} & b_{22}\mathbf{I} & \cdots & b_{2n}\mathbf{I} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}\mathbf{I} & b_{n2}\mathbf{I} & \cdots & b_{nn}\mathbf{I} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{B} + b_{11}\mathbf{I} & b_{12}\mathbf{I} & \cdots & b_{1n}\mathbf{I} \\ b_{21}\mathbf{I} & \mathbf{B} + b_{22}\mathbf{I} & \cdots & b_{2n}\mathbf{I} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}\mathbf{I} & b_{n2}\mathbf{I} & \cdots & \mathbf{B} + b_{nn}\mathbf{I} \end{bmatrix}.\end{aligned} \tag{4.52}$$

By assumption, the diagonal elements of $\mathbf{B}$ are positive, therefore, the diagonal elements of $\mathcal{B}$ are also positive.

Now, selecting $\mathbf{x} = \delta_i$, the $i$-th column of the identity matrix $\mathbf{I} \in \mathbb{R}^{n^2}$, it is easy to see that

$$\mathbf{y} = \mathcal{B}\mathbf{x} = \mathbf{b}_1 \tag{4.53}$$

where $\mathbf{b}_1$ is the first column of $\mathcal{B}$. Clearly, for $i = 1$, $x_1 > 0$ $(= 1)$; and $y_1 = 2b_{11}$ $(> 0$ by assumption). Therefore $x_1 y_1 > 0$, proving the result. ■

# 5. INTERVAL ROUTH-HURWITZ ARRAY AND STABILIZATION OF UNCERTAIN SYSTEMS

## 5.1. Introduction

Analysis and design of systems whose parameters are known to lie within a range (rather than having an exact value) has received considerable attention over the last decade. Various analysis and design techniques used by control systems engineers are essentially meant for application to a "nominal" model. The resulting design is said to be robust if the system performs within acceptable limits in the face of significant parameters variations and model uncertainties. There is a huge amount of literature that addresses the problems of stability and stabilization of interval polynomials. The first paper that treated the problem from a true interval point of view was the landmark paper by Kharitonov [22] regarding stability of polynomials whose coefficients are independent intervals. Later several researchers extended the result to more specialized interval polynomials, where the coefficients could have linearly coupled uncertainties. Notably, the works of Barmish [4], Keel *et al.*, [21], Saeki [36] and Zhou and Khargonekar [45], as well as the references therein provide a list of various efforts in this direction.

In this section, we define and develop the *interval Routh-Hurwitz array* [25] for a transfer function whose denominator is a polynomial with interval coefficients. Based

on the interval Routh-Hurwitz ($\mathcal{RH}$) array, we will present a procedure for finding a state feedback vector that will stabilize the entire interval system.

## 5.2. Interval Routh-Hurwitz Arrays

In this Section, we will develop the principles on which the interval $\mathcal{RH}$ arrays are based. The results regarding the $\mathcal{RH}$ arrays for the nominal systems are extended to the case of polynomials with interval coefficients.

### 5.2.1. PRELIMINARIES

The starting point for the subsequent analysis in this chapter is the description of a linear system (single input, single output) is in a differential equation of the following form:

$$\left[\frac{d^n}{dt^n} + [\,\check{a}_{n-1},\ \hat{a}_{n-1}\,]\frac{d^{n-1}}{dt^{n-1}} + [\,\check{a}_{n-2},\ \hat{a}_{n-2}\,]\frac{d^{n-2}}{dt^{n-2}} + \cdots + [\,\check{a}_1,\ \hat{a}_1\,]\frac{d}{dt} + [\,\check{a}_0,\ \hat{a}_0\,]\right] y(t)$$
$$= \left[c_m \frac{d^m}{dt^m} + c_{m-1}\frac{d^{m-1}}{dt^{m-2}} + \cdots + c_{m-2}\frac{d^{m-2}}{dt^{m-2}} + \cdots + c_1\frac{d}{dt} + c_0\right] u(t) \qquad (5.1)$$

whose transfer function is given by

$$G(s) = \frac{c_m s^m + c_{m-1}s^{m-1} + \cdots + c_1 s + c_0}{s^n + [\,\check{a}_{n-1},\ \hat{a}_{n-1}\,]s^{n-1} + \cdots + [\,\check{a}_1,\ \hat{a}_1\,]s + [\,\check{a}_0,\ \hat{a}_0\,]}. \qquad (5.2)$$

It should be pointed out that the coefficient of the highest derivative of $y(t)$ is assumed to be $[1,1]$, *i.e.*, the denominator polynomial of the system (5.2) is *monic*. This assumption is not necessary for developing the interval $\mathcal{RH}$ array; however, it is required for stabilization of (5.1) by state feedback. Note that such an assumption is not necessary for nominal systems because the polynomial can always be made monic by dividing it with the coefficient of the highest power of the derivative. This

operation cannot be performed on the interval polynomials because a multiplicative inverse is not defined for interval arithmetic. It will be assumed that the given system is controllable for all variations in the parameters.

It is well known that any proper scalar rational function $G(s) = Q(s)/P(s)$, where $P(s)$ is a monic polynomial, can be written in a controllable but not necessarily observable state space realization given by [19]:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & -p_{n-2} & -p_{n-1} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \tag{5.3}$$

$$y(t) = \begin{bmatrix} q_0 & q_1 & q_2 & \cdots & q_{n-2} & q_{n-1} \end{bmatrix} \mathbf{x}(t) + du(t) \tag{5.4}$$

where $p_i$ and $q_i$ are the coefficients of the $i$-th power of $s$ in the denominator and the numerator polynomials, respectively.

Typically, an interval $\mathcal{RH}$ array for a scalar system (e.g., 5-th order system) with the denominator polynomial having interval coefficients as shown below

$$\begin{aligned} d[s] = {} & [1,1]s^5 + [\,\check{a}_{2,1},\ \hat{a}_{2,1}\,]s^4 + [\,\check{a}_{1,2},\ \hat{a}_{1,2}\,]s^3 \\ & + [\,\check{a}_{2,2},\ \hat{a}_{2,2}\,]s^2 + [\,\check{a}_{1,3},\ \hat{a}_{1,3}\,]s^1 + [\,\check{a}_{2,3},\ \hat{a}_{2,3}\,], \end{aligned} \tag{5.5}$$

will have the following form:

| | | | |
|---|---|---|---|
| $s^5$ | $[1,1]$ | $[\,\check{a}_{1,2},\ \hat{a}_{1,2}\,]$ | $[\,\check{a}_{1,3},\ \hat{a}_{1,3}\,]$ |
| $s^4$ | $[\,\check{a}_{2,1},\ \hat{a}_{2,1}\,]$ | $[\,\check{a}_{2,3},\ \hat{a}_{2,3}\,]$ | $[\,\check{a}_{2,3},\ \hat{a}_{2,3}\,]$ |
| $s^3$ | $[\,\check{a}_{3,1},\ \hat{a}_{3,1}\,]$ | $[\,\check{a}_{3,2},\ \hat{a}_{3,2}\,]$ | |
| $s^2$ | $[\,\check{a}_{4,1},\ \hat{a}_{4,1}\,]$ | $[\,\check{a}_{4,2},\ \hat{a}_{4,2}\,]$ | |
| $s^1$ | $[\,\check{a}_{5,1},\ \hat{a}_{5,1}\,]$ | | |
| $s^0$ | $[\,\check{a}_{6,1},\ \hat{a}_{6,1}\,]$ | | |

where $\check{a}_{i,j}$ and $\hat{a}_{i,j}$ denote the lower and upper bounds of various interval coefficients obtained following the usual rules of setting up the $\mathcal{RH}$ array for polynomials with

constant coefficients. Specifically,

$$
\begin{aligned}
[\, \check{a}_{3,1},\ \hat{a}_{3,1}\, ] &= [[\, \check{a}_{2,1},\ \hat{a}_{2,1}\, ][\, \check{a}_{1,2},\ \hat{a}_{1,2}\, ] - [1,1][\, \check{a}_{2,2},\ \hat{a}_{2,2}\, ]]/[\, \check{a}_{2,1},\ \hat{a}_{2,1}\, ] \\
[\, \check{a}_{3,2},\ \hat{a}_{3,2}\, ] &= [[\, \check{a}_{2,1},\ \hat{a}_{2,1}\, ][\, \check{a}_{1,3},\ \hat{a}_{1,3}\, ] - [1,1][\, \check{a}_{2,3},\ \hat{a}_{2,3}\, ]]/[\, \check{a}_{2,1},\ \hat{a}_{2,1}\, ] \\
[\, \check{a}_{4,1},\ \hat{a}_{4,1}\, ] &= [[\, \check{a}_{3,1},\ \hat{a}_{3,1}\, ][\, \check{a}_{2,2},\ \hat{a}_{2,2}\, ] - [\, \check{a}_{2,1},\ \hat{a}_{2,1}\, ][\, \check{a}_{3,2},\ \hat{a}_{3,2}\, ]]/[\, \check{a}_{3,1},\ \hat{a}_{3,1}\, ] \\
[\, \check{a}_{4,2},\ \hat{a}_{4,2}\, ] &= [[\, \check{a}_{3,1},\ \hat{a}_{3,1}\, ][\, \check{a}_{2,3},\ \hat{a}_{2,3}\, ]]/[\, \check{a}_{3,1},\ \hat{a}_{3,1}\, ] \\
[\, \check{a}_{5,1},\ \hat{a}_{5,1}\, ] &= [[\, \check{a}_{4,1},\ \hat{a}_{4,1}\, ][\, \check{a}_{3,2},\ \hat{a}_{3,2}\, ] - [\, \check{a}_{3,1},\ \hat{a}_{3,1}\, ][\, \check{a}_{4,2},\ \hat{a}_{4,2}\, ]]/[\, \check{a}_{4,1},\ \hat{a}_{4,1}\, ] \\
[\, \check{a}_{6,1},\ \hat{a}_{6,1}\, ] &= [[\, \check{a}_{5,1},\ \hat{a}_{5,1}\, ][\, \check{a}_{4,2},\ \hat{a}_{4,2}\, ]]/[\, \check{a}_{5,1},\ \hat{a}_{5,1}\, ] \qquad (5.6)
\end{aligned}
$$

The only modification in the above array compared to the $\mathcal{RH}$ arrays for nominal polynomials is that to compute the elements of various rows, we use the principles of interval arithmetic following the operations reviewed in CHAPTER 2.

It was shown by Krishnamurthy [23] that for a stable system, the elements of any row of the $\mathcal{RH}$ array are of the same sign. Further, for the given system, if any element of a particular row is found to be of different sign, it can be concluded that the system is unstable and has one or more poles in the right half plane. In the next few paragraphs, we establish a similar result for the interval system by examining the elements of the interval $\mathcal{RH}$ array.

### 5.2.2. IMPLICATIONS OF INTERVAL $\mathcal{RH}$ ARRAYS

It was shown in [23], that given a *stable* nominal polynomial

$$F(s) = a_{11}s^n + a_{21}s^{n-1} + a_{12}s^{n-2} + \cdots \qquad a_{i,j} > 0 \qquad (5.7)$$

such that all elements of the first columns of the $\mathcal{RH}$ array are positive, it can be concluded that

**Theorem 5.1**: *For a stable polynomial, the necessary and sufficient condition of the $\mathcal{RH}$ criterion implies that all the elements of the $\mathcal{RH}$ array are of the same sign.*

PROOF: See Appendix 5.A.

For an $n$-th order polynomial, the following $\mathcal{RH}$ array can be set up:

$$
\begin{array}{c|ccccc}
s^n & a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,j_m} \\
s^{n-1} & a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,j_m} \\
\hline
s^{n-2} & a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,j_m} \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
s^{n-i+1} & a_{i,1} & a_{i,2} & a_{i,3} & \cdots & a_{i,j_m} \\
\vdots & \vdots & \vdots & \iddots & & \iddots \\
s^3 & a_{n-2,1} & a_{n-2,2} & & & \\
s^2 & a_{n-1,1} & a_{n-1,2} & & & \\
s^1 & a_{n,1} & & & & \\
s^0 & a_{n+1,1} & & & &
\end{array}
$$

where $j_m = \lfloor (n+3-i)/2 \rfloor$ is the largest integer value of the column with a non-zero entry in the $i$-th row. Note that the above notation is for both odd and even degree polynomials. However, the results outlined below will be for odd degree polynomials only. Parallel results can be easily obtained for even degree polynomials.

For a stable polynomial, the following three properties hold:

**Property 1.** All elements in the first column of the array are positive

$$a_{i,1} > 0, \quad i \;=\; 1, 2, \ldots, n+1 \tag{5.8}$$

**Property 2.** The number of coefficients in alternate rows is reduced by one,

$$a_{i,j} \;=\; 0, \quad i \;=\; 1, 2, \ldots, n+1, \;\; j > \lfloor (n+3-i)/2 \rfloor \tag{5.9}$$

where $\lfloor(\cdot)\rfloor$ denotes the integer part of the number.

**Property 3.** The constant term appears as the last element of alternate rows, *i.e.*

$$a_{n+3-2k,k} = \text{constant term}, \quad k = 1,2,\ldots,\lfloor (n+2)/2) \rfloor . \tag{5.10}$$

To extend the result to the interval $\mathcal{RH}$ arrays, consider the following odd powered interval monic polynomial:

$$F(s) = [1,1]s^n + [\,\check{a}_{21},\ \hat{a}_{21}\,]s^{n-1} + [\,\check{a}_{12},\ \hat{a}_{12}\,]s^{n-2} + \cdots, \quad [\,\check{a}_{i,j},\ \hat{a}_{i,j}\,] > 0 \tag{5.11}$$

The corresponding interval $\mathcal{RH}$ array is given by

$$\begin{array}{c|lllll}
s^n & [1,1] & [\,\check{a}_{1,2},\ \hat{a}_{1,2}\,] & [\,\check{a}_{1,3},\ \hat{a}_{1,3}\,] & \cdots & [\,\check{a}_{1,j_m},\ \hat{a}_{1,j_m}\,] \\
s^{n-1} & [\,\check{a}_{2,1},\ \hat{a}_{2,1}\,] & [\,\check{a}_{2,2},\ \hat{a}_{2,2}\,] & [\,\check{a}_{2,3},\ \hat{a}_{2,3}\,] & \cdots & [\,\check{a}_{2,j_m},\ \hat{a}_{2,j_m}\,] \\
\hline
s^{n-2} & [\,\check{a}_{3,1},\ \hat{a}_{3,1}\,] & [\,\check{a}_{3,2},\ \hat{a}_{3,2}\,] & [\,\check{a}_{3,3},\ \hat{a}_{3,3}\,] & \cdots & [\,\check{a}_{3,j_m},\ \hat{a}_{3,j_m}\,] \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
s^{n-i+1} & [\,\check{a}_{i,1},\ \hat{a}_{i,1}\,] & [\,\check{a}_{i,2},\ \hat{a}_{i,2}\,] & [\,\check{a}_{i,3},\ \hat{a}_{i,3}\,] & \cdots & [\,\check{a}_{i,j_m},\ \hat{a}_{i,j_m}\,] \\
\vdots & \vdots & \vdots & \iddots & & \iddots \\
s^3 & [\,\check{a}_{n-2,1},\ \hat{a}_{n-2,1}\,] & [\,\check{a}_{n-2,2},\ \hat{a}_{n-2,2}\,] & & & \\
s^2 & [\,\check{a}_{n-1,1},\ \hat{a}_{n-1,1}\,] & [\,\check{a}_{n-1,2},\ \hat{a}_{n-1,2}\,] & & & \\
s^1 & [\,\check{a}_{n,1},\ \hat{a}_{n,1}\,] & & & & \\
s^0 & [\,\check{a}_{n+1,1},\ \hat{a}_{n+1,1}\,] & & & &
\end{array}$$

where all elements of the array have been calculated using the principles of interval arithmetic.

Unlike the $\mathcal{RH}$ array for nominal polynomials, the PROPERTY 3 does not hold for interval $\mathcal{RH}$ arrays. However, the following is true

$$[\,\check{a}_{n+1,1},\ \hat{a}_{n+1,1}\,] \supseteq [\,\check{a}_{n-1,2},\ \hat{a}_{n-1,2}\,] \supseteq [\,\check{a}_{n-3,3},\ \hat{a}_{n-3,3}\,] \supseteq \ \cdots\ \supseteq [\,\check{a}_{2,j_m},\ \hat{a}_{2,j_m}\,]. \tag{5.12}$$

This can be shown trivially following the sequence of operations performed to obtain the interval $\mathcal{RH}$ array. This property will be referred to as PROPERTY $\tilde{3}$.

Clearly by PROPERTY 1, $[\, \check{a}_{n+1,1},\ \hat{a}_{n+1,1}\, ]$ is a positive interval, therefore, the intervals $[\, \check{a}_{n-1,2},\ \hat{a}_{n-1,2}\, ]\ [\, \check{a}_{n-3,3},\ \hat{a}_{n-3,3}\, ] \cdots [\, \check{a}_{2,j_m},\ \hat{a}_{2,j_m}\, ]$ are also positive.

Following the same reasoning as we did for the case of nominal polynomials, we next develop the implications of the interval $\mathcal{RH}$ array. Obviously, from PROPERTY 1, the $(n+1)$-th and $n$-th rows have positive intervals. Also, from PROPERTIES 1, $\tilde{3}$, the interval in $(n-1)$-th row are positive. In $(n-2)$-th row, to show that the second interval is positive, we proceed as follows. By definition,

$$[\check{a}_{n,1}, \hat{a}_{n,1}] = \frac{[\, \check{a}_{n-1,1},\ \hat{a}_{n-1,1}\, ][\, \check{a}_{n-2,2},\ \hat{a}_{n-2,2}\, ] - [\, \check{a}_{n-2,1},\ \hat{a}_{n-2,1}\, ][\, \check{a}_{n-1,2},\ \hat{a}_{n-1,2}\, ]}{[\, \check{a}_{n-1,1},\ \hat{a}_{n-1,1}\, ]} \tag{5.13}$$

performing the inverse interval operations, we have

$$\frac{[\, \check{a}_{n,1},\ \hat{a}_{n,1}\, ][\, \check{a}_{n-1,1},\ \hat{a}_{n-1,1}\, ] + [\, \check{a}_{n-2,1},\ \hat{a}_{n-2,1}\, ][\, \check{a}_{n-1,2},\ \hat{a}_{n-1,2}\, ]}{[\, \check{a}_{n-1,1},\ \hat{a}_{n-1,1}\, ]} \supseteq [\, \check{a}_{n-2,2},\ \hat{a}_{n-2,2}\, ]. \tag{5.14}$$

Each interval of the expression on the left-hand side of (5.14) is positive, therefore, $[\, \check{a}_{n-2,2},\ \hat{a}_{n-2,2}\, ]$ is contained in a positive interval.

In the $(n-3)$-th row, there are three elements, from PROPERTIES 1,$\tilde{3}$, the first and the last elements are positive intervals. To show that the second element is also positive, we know that

$$[\, \check{a}_{n-1,1},\ \hat{a}_{n-1,1}\, ] = \frac{[\, \check{a}_{n-2,1},\ \hat{a}_{n-2,1}\, ][\, \check{a}_{n-3,2},\ \hat{a}_{n-3,2}\, ] - [\, \check{a}_{n-3,1},\ \hat{a}_{n-3,1}\, ][\, \check{a}_{n-2,2},\ \hat{a}_{n-2,2}\, ]}{[\, \check{a}_{n-2,1},\ \hat{a}_{n-2,1}\, ]}. \tag{5.15}$$

From where, performing the inverse operations, we get

$$\frac{[\, \check{a}_{n-1,1},\ \hat{a}_{n-1,1}\, ][\, \check{a}_{n-2,1},\ \hat{a}_{n-2,1}\, ] + [\, \check{a}_{n-3,1},\ \hat{a}_{n-3,1}\, ][\, \check{a}_{n-2,2},\ \hat{a}_{n-2,2}\, ]}{[\, \check{a}_{n-2,1},\ \hat{a}_{n-2,1}\, ]}$$

$$\supseteq [\, \check{a}_{n-3,2},\ \hat{a}_{n-3,2} \,] \tag{5.16}$$

Again, since each element of the on the right hand side of (5.16) is positive, $a_{n-3,2}$ is also positive.

Continuing in this manner, we show that the elements of the $i$-th row are positive intervals. The first element is positive by PROPERTY 1. The $j$-th element of the $i$-th row is given by

$$[\, \check{a}_{i+2,j-1},\ \hat{a}_{i+2,j-1} \,] = \frac{[\, \check{a}_{i+1,i},\ \hat{a}_{i+1,i} \,]a_{i,j} - [\, \check{a}_{i,1},\ \hat{a}_{i,1} \,][\, \check{a}_{i+1,j},\ \hat{a}_{i+1,j} \,]}{[\, \check{a}_{i+1,1},\ \hat{a}_{i+1,1} \,]}. \tag{5.17}$$

To get an inclusion for $[\, \check{a}_{i,j},\ \hat{a}_{i,j} \,]$ from (5.17), we perform the inverse operation to get

$$\frac{[\, \check{a}_{i+1,1},\ \hat{a}_{i+1,1} \,][\, \check{a}_{i+2,j-1},\ \hat{a}_{i+2,j-1} \,] + [\, \check{a}_{i,1},\ \hat{a}_{i,1} \,][\, \check{a}_{i+1,j},\ \hat{a}_{i+1,j} \,]}{[\, \check{a}_{i+1,1},\ \hat{a}_{i+1,1} \,]} \supseteq [\, \check{a}_{i,j},\ \hat{a}_{i,j} \,]. \tag{5.18}$$

Since all of the intervals on the left-hand side are either positive by PROPERTY 1 or have been proved to be positive in the previous cycle, the interval on the right hand side in (5.18) is positive.

Further, for even $i$, the last element $[\, \check{a}_{i,j_m},\ \hat{a}_{i,j_m} \,]$ is included in the interval $[\, \check{a}_{i+1,1},\ \hat{a}_{i+1,1} \,]$ and therefore it is positive. If $i$ is odd, then

$$\frac{[\, \check{a}_{i+2,j_m},\ \hat{a}_{i+2,j_m} \,][\, \check{a}_{i+1,1},\ \hat{a}_{i+1,1} \,] + [\, \check{a}_{i+1,j_m},\ \hat{a}_{i+1,j_m} \,][\, \check{a}_{i,1},\ \hat{a}_{i,1} \,]}{[\, \check{a}_{i+1,1},\ \hat{a}_{i+1,1} \,]} \supseteq [\, \check{a}_{i,j_m},\ \hat{a}_{i,j_m} \,], \tag{5.19}$$

which is a positive interval.

Therefore for a stable interval polynomial with positive first column of the interval $\mathcal{RH}$ array, every element of the array is positive. Similar results can be obtained when $n$ is an even ordered interval polynomial.

It should be pointed out that unlike the classical $\mathcal{RH}$ array, the interval $\mathcal{RH}$ array provide only a sufficient condition for stability. This is due to the fact that in performing the interval operations, the intervals are implicitly overbounded.

**Example 5.1**: To illustrate the above point, consider the polynomial given by

$$F(s) \;=\; s^3 + [\,\check{a}_{21},\ \hat{a}_{21}\,]s^2 + [\,\check{a}_{12},\ \hat{a}_{12}\,]s^1 + 1.$$

where $[\,\check{a}_{12},\ \hat{a}_{12}\,] = [4,6]$ and $[\,\check{a}_{21},\ \hat{a}_{21}\,] = [2,3]$. Now, treating the two intervals as variable coefficients $r_{21}$ and $r_{12}$, we get the following nominal $\mathcal{RH}$ array:

$$\begin{array}{c|ll}
s^3 & 1 & r_{12} \\
s^2 & r_{21} & 1 \\
\hline
s^1 & \frac{r_{12}r_{21}-1}{r_{21}} & 0 \\
s^0 & 1 & 0
\end{array}$$

On substituting the values of $r_{12}$ and $r_{21}$ for maximum allowable variation in the two uncertain coefficients, we get the following array:

$$\begin{array}{c|ll}
s^3 & 1 & [4,6] \\
s^2 & [2,3] & 1 \\
\hline
s^1 & [3.5, 5.6667] & 0 \\
s^0 & 1 & 0
\end{array}$$

On the contrary, if the intervals were allowed to "grow" as the array was computed, the corresponding interval $\mathcal{RH}$ array will be

$$\begin{array}{c|ll}
s^3 & 1 & [4,6] \\
s^2 & [2,3] & 1 \\
\hline
s^1 & [2.3333, 8.5] & 0 \\
s^0 & [0.2745, 3.6429] & 0
\end{array}$$

It is clear from the above example, that using interval arithmetic, the results will be overbounded.

### 5.2.3. REMOVAL OF CONSERVATISM

The example (EXAMPLE 5.1) presented in the previous section provides a possible way to remove the conservatism in the analysis procedure. Next, we formalize one possible way for removing the conservatism in developing the interval $\mathcal{RH}$ arrays.

To get an intuitive understanding of the procedure, let us consider the element $x_{31}$ in the first column of $s^1$ row from EXAMPLE 5.1 given by

$$x_{31} = \frac{r_{12}r_{21} - 1}{r_{21}} \tag{5.20}$$

Notice that using interval arithmetic, $x_{13}$ is determined as

$$\begin{aligned}
[\, \check{x}_{13},\ \hat{x}_{13} \,] &= \frac{r_{12}r_{21} - 1}{r_{21}} \\
&= \frac{[4,6][2,3] \ - \ [1,1]}{[2,3]} \\
&= [7,17][1/3,1/2] \\
&= [2.3333, 8.5].
\end{aligned}$$

The element $r_{21}$ occurs both in the numerator and the denominator. The interval inclusion obtained above does not account for this fact. It treats each interval as independent of the other intervals and in the process overbounds the interval.

If, on the other hand, we account for the fact that $r_{21}$ can only assume one value both in the numerator as well as the denominator, then the interval $[\, \check{x}_{13},\ \hat{x}_{13} \,]$ $(= [3.5, 5.6667])$ so obtained is the maximum variation in light of multiple occurrence of $r_{21}$. This fact can be utilized to remove the conservatism from the interval $\mathcal{RH}$ arrays.

To enable us to obtain such minimum and maximum values for each interval in the first column of the interval $\mathcal{RH}$ array, we need a numerical algorithm that computes

the extrema. Fortunately, Moore [27] and Skelboe [39] provide the necessary details for developing an efficient algorithm for obtaining the extrema. A detailed account of these techniques is presented by Ratschek and Rokne in [34]. In the sequel, we present the highlights of the algorithm. The notation that we have adopted is similar to the one in [34].

The problem that we need to address is that of the *global unconstrained optimization* which can be stated as follows: Let $\mathbb{R}$ be the set of reals, let $\mathbf{X} \subseteq \mathbb{R}^n$ be a compact right parallelepiped parallel to the axes (denoted by a $[\!]$), $f : \mathbf{X} \to \mathbb{R}$ any function and $[\!] f(\mathbf{X})$ the range of the function $f$ over $\mathbf{X}$, *i.e.*

$$[\!] f(\mathbf{X}) = \{ f(\mathbf{x}) \ : \ \mathbf{x} \in \mathbf{X} \}. \tag{5.21}$$

The *global minimum* (if it exists in $\mathbb{R}$), defined as infimum $\left([\!] f(\mathbf{X})\right)$ is denoted by $\check{f}$. The *global maximum* (if it exists in $\mathbb{R}$), defined as supremum $\left([\!] f(\mathbf{X})\right)$ is denoted by $\hat{f}$. Note that infimum and supremum are being computed instead of minimum and maximum because of lack of assumption of continuity on the function. Mathematically, the problem defined above can be stated as

$$\text{minimize } f(\mathbf{x}) \text{ subject to } \mathbf{x} \in \mathbf{X} \tag{5.22a}$$

$$\text{maximize } f(\mathbf{x}) \text{ subject to } \mathbf{x} \in \mathbf{X}. \tag{5.22b}$$

In the following paragraphs, only minimization will be considered. The procedure for maximization is identical except for a change in sign of $f$.

Given the bounded function $f$, the domain $[\ \check{\mathbf{X}},\ \hat{\mathbf{X}}\ ] \in \mathbb{I}\mathbb{R}^n$ and an initial choice of inclusion function $F : \mathbb{I}\mathbb{R}(\mathbf{X}) \to \mathbb{I}\mathbb{R}$ of $f(\mathbf{x})$, the algorithm works by splitting up the domain $\mathbf{X}$ in each step, into subboxes of not necessarily the same size. The search for

$\check{f}$ is done in the subboxes. A direct search would be computationally very demanding. hence a ***branching principle*** is used. At each iteration, the search is continued in a box $\mathbf{Y}$, where $f$ has the smallest lower bound $y$, because the chances of finding $\check{f}$ are the best in this box.

**Algorithm 5.1:** GLOBAL UNCONSTRAINED OPTIMIZATION

**Set** $\mathbf{Y} := \mathbf{X}$

**Calculate** $F(\mathbf{Y})$

**Set** $\mathbf{y} :=$ min $F(\mathbf{Y})$

**Initialize** list $\mathcal{L} := ((\mathbf{Y}, \mathbf{y}))$

**Iterate:**

**Select** a coordinate direction $k$ parallel to which $\mathbf{Y} = \mathbf{Y}_1 \times \cdots \times \mathbf{Y}_n$ has an edge of maximum length i.e. $k \in \{i : w(\mathbf{Y}) = w(\mathbf{Y}_i)\}$

**Bisect** $\mathbf{Y}$ normal to direction $k$ obtaining boxes $\mathbf{V}_1$ and $\mathbf{V}_2$ such that $\mathbf{Y} = \mathbf{V}_1 \cup \mathbf{V}_2$

**Calculate** $F(\mathbf{V}_1)$ and $F(\mathbf{V}_1)$

**Set** $\mathbf{v}_i = F_{lb}(\mathbf{V}_i)$, $i = 1, 2$

**Delete** $(\mathbf{Y}, \mathbf{y})$ from the list $\mathcal{L}$

**Include** $(\mathbf{V}_1, \mathbf{v}_1)$ and $(\mathbf{V}_2, \mathbf{v}_2)$ in the list such that second members of all pairs of list do not decrease

**Denote** the first pair of the list by $(\mathbf{Y}, \mathbf{y})$

**If** $w(F(\mathbf{Y}_k)) < \epsilon$ then **terminate** else **iterate**

The detailed implementation of the algorithm as well as further details regarding the convergence and termination criteria can be found in [34]. It suffices to say that the

necessary tools for obtaining the correct bounds on the intervals in the first column of the interval $\mathcal{RH}$ array are available. It is also worth mentioning that several software packages that implement machine interval arithmetic exist. Some of the commonly used ones are TRIPLEX-ALGOL-60, PASCAL-SC, FORTRAN-SC, ACRITH and ARITHMOS.

Using above algorithm, we can make the condition on the interval $\mathcal{RH}$ array both necessary and sufficient. The modified procedure would require the determination of the elements of the first column of the array *symbolically* and then application of above algorithm will provide us with the correct lower and upper bounds of each element in the first column of the interval $\mathcal{RH}$ array. For symbolic computation software packages like MATHEMATICA can be used.

## 5.3. STABILITY OF SYSTEMS WITH DEPENDENT UNCERTAINTIES

Unlike systems described in earlier section, where each coefficient has variations that are totally independent of the rest, many systems exhibit dependent uncertainties [4], [21], [36] [45]. Specifically, each coefficient may have several uncertain parameters. Further, these parameters may exhibit some functional behavior. In this section, we will extend the results of SECTION 5.2.3 to systems with dependent uncertainties. It will be shown that in view of the previous section, the stability of interval polynomials with dependent uncertainties can be determined in a simple manner. Note that several papers deal with the above issue, however, the techniques tend to be either too complicated or computationally extremely demanding. In the next few paragraphs we show that the procedure outlined in the previous section could be easily applied to study the stability properties of polynomials whose coefficients exhibit linear and

non-linear dependent uncertainties. Formal proofs of the material presented in the sequel have not been developed. Hence, only examples have been used to illustrate the concepts.

Let us denote the characteristic polynomial of the systems with coupled uncertainties as

$$d[s] = s^n + \sum_{i=0}^{n-1}[f_i(\mathbf{r})]s^i \tag{5.23}$$

where $f_i(\mathbf{r})$ denotes the coefficient of the $i$-th power of $s$ and $\mathbf{r}$ is the vector of uncertain terms.

We discuss three possible ways to treat the situation discussed above.

1. The first approach could be to overbound the various coefficients using interval arithmetic directly, and use interval $\mathcal{RH}$ criterion. As expected this would lead to conservative results.

**Example 5.2**: To illustrate this point, consider the polynomial described by

$$d[s] = s^3 + [r_1 + r_2]s^2 + [r_1 r_2]s + [r_2^2],$$

where, $r_1 \in [0.74, 0.76]$ and $r_2 \in [2.00, 2.50]$. The polynomial has two uncertain elements which appear as coupled uncertainties in various coefficients. On expanding the polynomial, the numerical values in various coefficients are given by

$$d[s] = s^3 + [2.74, 3.26]s^2 + [1.48, 1.9]s + [4, 6.25].$$

Now, using interval $\mathcal{RH}$ array, we have:

| | | |
|---|---|---|
| $s^3$ | 1 | [1.48, 1.90] |
| $s^2$ | [2.74, 3.26] | [4.00, 6.25] |
| $s^1$ | [−0.8010, .8007] | – |
| $s^0$ | undefined | 0 |

Note that in the above array, the first element in the third row includes zero, since the elements of the subsequent rows would require division by this element, we cannot proceed any further. Clearly the criterion suggests a sign change in the first column and hence instability.

2. The second possibility is to verify whether the expanded polynomial meets Kharitonov's criterion. This entails checking the stability of the following four polynomials:

$$s^3 + 3.26s^2 + 1.48s + 4.00$$

$$s^3 + 2.74s^2 + 1.90s + 6.25$$

$$s^3 + 3.26s^2 + 1.90s + 4.00$$

$$s^3 + 2.74s^2 + 1.48s + 6.25.$$

While the first and third polynomials are stable, the second and the fourth polynomials are unstable, hence use of Kharitonov criterion on the expanded polynomial also indicates instability.

3. Finally, the third approach would be to use the Global Unconstrained Optimization on the elements of the symbolic $\mathcal{RH}$ array and infer stability from there. The array takes the following form:

$$\begin{array}{c|ll}
s^3 & 1 & r_1 r_2 \\
s^2 & r_1 + r_2 & r_2^2 \\
\hline
s^1 & r_2\left(r_1 - \frac{r_2}{r_1+r_2}\right) & - \\
s^0 & 1 & -
\end{array}$$

Now, using unconstrained optimization, the first element in the third row is found to be [0.0201, 0.0884], which is positive and hence the polynomial with dependent coefficients is stable.

The technique is simple to apply because the optimization technique is commercially available in packages that implement interval arithmetic.

## 5.4. Feedback Stabilization of Uncertain Systems

This section presents results on the use of the techniques developed above in robust synthesis of feedback control systems. The results presented in this section are for single input, single output systems only.

### 5.4.1. SYSTEMS WITH INDEPENDENT UNCERTAINTIES

In this section, we will show that by using the interval $\mathcal{RH}$ arrays, we can determine a state feedback vector $\mathbf{k}$ (when it exists) such that the closed loop "interval" system is always stable. For the purpose of illustration, we will develop all results for a fifth order system. The final results will be stated for the general order. It is assumed that the uncertainties in various coefficients are mutually independent. Further assume that the system is in the following canonical form:

$$\frac{d\mathbf{x}(t)}{dt} = \left[\frac{\mathbf{0} \mid \mathbf{I}_{n-1}}{[\mathbf{a}]^T}\right]\mathbf{x}(t) + \mathbf{b}u(t) \tag{5.24}$$

$$y(t) = \mathbf{cx}(t) \tag{5.25}$$

where

$$[\mathbf{a}]^T = \left[\, -[\,\check{a}_{2,3},\ \hat{a}_{2,3}\,] \ \ -[\,\check{a}_{1,3},\ \hat{a}_{1,3}\,] \ \ -[\,\check{a}_{2,2},\ \hat{a}_{2,2}\,] \ \ -[\,\check{a}_{1,2},\ \hat{a}_{1,2}\,] \ \ -[\,\check{a}_{2,1},\ \hat{a}_{2,1}\,]\,\right]$$

$$\mathbf{b}^T = \left[\, 0\ 0\ 0\ 0\ 1 \,\right]$$

$$\mathbf{c} = \left[\, c_{2,3}\ \ c_{1,3}\ \ c_{2,2}\ \ c_{1,2}\ \ c_{2,1} \,\right]$$

Define a state feedback vector $\mathbf{k}^T = [k_{2,3}\ \ k_{1,3}\ \ \cdots\ \ k_{2,1}]$. Then the state matrix

of the closed loop state space system under the feedback law

$$u(t) = v(t) - \mathbf{k}^T \mathbf{x}(t) \tag{5.26}$$

becomes

$$\mathbf{A}_{cl} = \left[\frac{\mathbf{0} \mid \mathbf{I}_{n-1}}{[\mathbf{a}(\mathbf{k})]^T}\right]$$

where $[\mathbf{a}(\mathbf{k})]$ is given by

$$[\mathbf{a}(\mathbf{k})] = \begin{bmatrix} -[\,\check{a}_{2,3},\ \hat{a}_{2,3}\,] - k_{2,3} \\ -[\,\check{a}_{1,3},\ \hat{a}_{1,3}\,] - k_{1,3} \\ -[\,\check{a}_{2,2},\ \hat{a}_{2,2}\,] - k_{2,2} \\ -[\,\check{a}_{1,2},\ \hat{a}_{1,2}\,] - k_{1,2} \\ -[\,\check{a}_{2,1},\ \hat{a}_{2,1}\,] - k_{2,1} \end{bmatrix}. \tag{5.27}$$

Since $\mathbf{A}_{cl}$ is in the companion form, its characteristic polynomial is given by

$$\begin{aligned} d_c[s] = {} & [1,1]s^5 + [\check{a}_{2,1} + k_{2,1}, \hat{a}_{2,1} + k_{2,1}]s^4 \\ & + [\check{a}_{1,2} + k_{1,2}, \hat{a}_{1,2} + k_{1,2}]s^3 + [\check{a}_{2,2} + k_{2,2}, \hat{a}_{2,2} + k_{2,2}]s^2 \\ & + [\check{a}_{1,3} + k_{1,3}, \hat{a}_{1,3} + k_{1,3}]s^1 + [\check{a}_{2,3} + k_{2,3}, \hat{a}_{2,3} + k_{2,3}]s^0, \end{aligned} \tag{5.28}$$

and the corresponding interval $\mathcal{RH}$ array is modified to

$$\begin{array}{c|lll} s^5 & [1,1] & [\check{a}_{1,2}(\mathbf{k}), \hat{a}_{1,2}(\mathbf{k})] & [\check{a}_{1,3}(\mathbf{k}), \hat{a}_{1,3}(\mathbf{k})] \\ s^4 & [\check{a}_{2,1}(\mathbf{k}), \hat{a}_{2,1}(\mathbf{k})] & [\check{a}_{2,2}(\mathbf{k}), \hat{a}_{2,2}(\mathbf{k})] & [\check{a}_{2,3}(\mathbf{k}), \hat{a}_{2,3}(\mathbf{k})] \\ \hline s^3 & [\check{a}_{3,1}(\mathbf{k}), \hat{a}_{3,1}(\mathbf{k})] & [\check{a}_{3,2}(\mathbf{k}), \hat{a}_{3,2}(\mathbf{k})] & \\ s^2 & [\check{a}_{4,1}(\mathbf{k}), \hat{a}_{4,1}(\mathbf{k})] & [\check{a}_{4,2}(\mathbf{k}), \hat{a}_{4,2}(\mathbf{k})] & \\ s^1 & [\check{a}_{5,1}(\mathbf{k}), \hat{a}_{5,1}(\mathbf{k})] & & \\ s^0 & [\check{a}_{6,1}(\mathbf{k}), \hat{a}_{6,1}(\mathbf{k})] & & \end{array}$$

where $\check{a}_{i,j}(\mathbf{k})$ and $\hat{a}_{i,j}(\mathbf{k})$ are now functions of the elements of the state feedback vector $\mathbf{k}$. Now extending the property of classical $\mathcal{RH}$ arrays to interval $\mathcal{RH}$ arrays, we have the following result:

**Theorem 5.2** *A sufficient condition for all roots of the interval polynomial $d[s]$ to lie in the left half plane is that the intervals* $[\,\check{a}_{i,j}(\mathbf{k}),\ \hat{a}_{i,j}(\mathbf{k})\,],\ i\ =\ 1,2,\ldots,n+1,\ \forall\ j,$ *of the interval $\mathcal{RH}$ array have the same sign.*

To solve for the elements of $\mathbf{k}$, assuming that the coefficient of the highest power of $s$ is $[1,1]$, we set up the following $n$ non-linear algebraic inequalities in $n$ unknowns:

$$
\begin{aligned}
\check{a}_{2,1}(\mathbf{k}) &> 0 \\
\check{a}_{3,1}(\mathbf{k}) &> 0 \\
&\ \vdots \\
\check{a}_{n+1,1}(\mathbf{k}) &> 0.
\end{aligned}
\tag{5.29}
$$

It is fairly straightforward to see that if a solution to above simultaneous inequalities exists, it will provide us with an $n$ dimensional coefficient space, any element from which would ensure the stability of the interval system.

Note that by the definition of an interval, it is assured that the first element of the interval is smaller than the second. Hence, if the lower extreme of the intervals in the first columns have positive sign, then the upper extremes are guaranteed to have the same sign. Further, from the results in SECTION 3.2, every element of each row will have the same sign, satisfying the condition in [23]. The above design procedure is illustrated by means of a 3rd order system.

**Example 5.3**: Assume that the denominator polynomial of the interval system is

$$d[s]\ =\ [1,1]s^3\ +\ [-2,1]s^2\ +\ [2,3]s\ +\ [0,2].$$

Clearly, the interval polynomial is not stable. We need to find a state feedback vector

$\mathbf{k}^T = [k_0 \;\; k_1 \;\; k_2]$ such that

$$d[s] = [1,1]s^3 + [-2+k_2, 1+k_2]s^2 + [2+k_1, 3+k_1]s + [0+k_0, 2+k_0]$$

is stable for the entire interval. Using the interval $\mathcal{RH}$ array, and a fair amount of painful symbolic manipulation, one gets the following set of non-linear algebraic inequalities:

$$-2 + k_2 > 0$$

$$(2+k_1)(-2+k_2) - (2+k_0) > 0$$

$$k_0(-2+k_2)((2+k_1)(-2+k_2) - (2+k_0)) > 0$$

that define the region from which the parameters of $\mathbf{k}$ can be obtained to stabilize the system. One possible stabilizing feedback vector is obtained as $\mathbf{k} = [1 \;\; 2 \;\; 3]$. The corresponding closed loop characteristic polynomial is given by

$$d_c[s] = [1,1]s^3 + [1,4]s^2 + [4,5]s + [1,3].$$

It can be readily verified that $d_c[s]$ is stable for entire range of interval coefficients.

Since the stability condition presented above is only sufficient, clearly the design so obtained will be conservative in nature. However, the usefulness of employing interval arithmetic is self-evident in that it provides us with a solution (when it exists) in a fairly straightforward manner.

It is easy to see that the above results can be used for simultaneous stabilization by means of state feedback. Assume that we are given $N$ plants with characteristic

polynomials

$$
\begin{aligned}
d^{[1]}(s) &= s^n + \sum_{i=0}^{n-1} a_i^{[1]} s^i \\
d^{[2]}(s) &= s^n + \sum_{i=0}^{n-1} a_i^{[2]} s^i \\
&\vdots \\
d^{[N]}(s) &= s^n + \sum_{i=0}^{n-1} a_i^{[N]} s^i.
\end{aligned} \tag{5.30}
$$

Assume further that the order of each plant is $n$. Then, we can easily obtain an interval plant defined as

$$
d[s] = s^n + \sum_{i=0}^{n-1} [\, \check{a}_i^{[j]},\ \hat{a}_i^{[k]}\, ] s^i, \tag{5.31}
$$

where $\check{a}_i^{[j]} = \min(a_i^{[j]})$ and $\hat{a}_i^{[j]} = \max(a_i^{[j]})$, $j = 1, 2, \ldots, N$. If a controller that can stabilize the above interval plant can be found then, clearly, it will stabilize all the plants in the above family.

## APPENDIX 5.A.

PROOF OF THEOREM 5.1: Although the proof appears in [23], an abbreviated version of the proof is included for the sake of completeness. For an $n$-th order polynomial, the following $\mathcal{RH}$ array can be set up:

$$
\begin{array}{c|lllll}
s^n & a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,j_m} \\
s^{n-1} & a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,j_m} \\
\hline
s^{n-2} & a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,j_m} \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
s^{n-i+1} & a_{i,1} & a_{i,2} & a_{i,3} & \cdots & a_{i,j_m} \\
\vdots & \vdots & \vdots & \iddots & & \iddots \\
s^3 & a_{n-2,1} & a_{n-2,2} & & & \\
s^2 & a_{n-1,1} & a_{n-1,2} & & & \\
s^1 & a_{n,1} & & & & \\
s^0 & a_{n+1,1} & & & &
\end{array}
$$

where $j_m = \lfloor (n+3-i)/2 \rfloor$ is the largest integer value of the column with a non-zero entry in the $i$-th row.

For a stable polynomial, the following three properties (repeated here for the sake of completeness) hold:

**Property 1.** All elements in the first column of the array are positive

$$a_{i,1} > 0, \quad i = 1, 2, \ldots, n+1 \tag{5.32}$$

**Property 2.** The number of coefficients in alternate rows is reduced by one,

$$a_{i,j} = 0, \quad i = 1, 2, \ldots, n+1, \; j > \lfloor (n+3-i)/2 \rfloor \tag{5.33}$$

where $\lfloor (\cdot) \rfloor$ denotes the integer part of the number.

**Property 3.** The constant term appears as the last element of alternate rows, *i.e.*

$$a_{n+3-2k,k} = \text{constant term}, \quad k = 1, 2, \ldots, \lfloor (n+2)/2) \rfloor. \tag{5.34}$$

Now, from PROPERTY 1, the $(n+1)$-th and $n$-th rows are clearly positive. From PROPERTIES 1, 3, the elements in $(n-1)$-th row are positive. In $(n-2)$-th row, to show that the second element is positive,

$$a_{n-2,2} = a_{n,1} + \frac{a_{n-2,1}}{a_{n-1,1}} a_{n-1,2}. \tag{5.35}$$

Since each element of the elements on the right hand side of (5.35) is positive, $a_{n-2,2}$ is also positive.

In the $(n-3)$-th row, there are three elements, from PROPERTIES 1,3 the first and the last elements are positive. To show that the the second element is also positive,

$$a_{n-3,2} = a_{n-1,1} + \frac{a_{n-3,1}}{a_{n-2,1}} a_{n-2,2}. \tag{5.36}$$

Again, since each element of the elements on the right hand side of (5.36) is positive $a_{n-3,2}$ is also positive.

Proceeding in a similar manner, consider the elements of the $i$-th row. The first element is positive by PROPERTY 1.

$$a_{i,2} = a_{i+2,1} + \frac{a_{i,1}}{a_{i+1,1}} a_{i+1,2} \tag{5.37}$$

where $a_{i+1,2}$ would have been proved to be positive from previous iteration, hence $a_{i,2}$ is positive. Taking the general term

$$a_{i,j} = a_{i+2,j-1} + \frac{a_{i,1}}{a_{i+1,1}} a_{i+1,j} \tag{5.38}$$

is positive by the same reasoning as above.

For even $i$, the last element $a_{i,j_m}$ is the same as the constant term and therefore it is positive. If $i$ is odd, then

$$a_{i,j_m} = a_{i+2,j_m} + \frac{a_{i,1}}{a_{i+1,1}} a_{i+1,j_m}. \tag{5.39}$$

Therefore, $a_{i,j_m}$ is also positive. Similar results hold when $n$ is even ordered. ■

# 6. RECOMMENDATIONS FOR FUTURE WORK

The results presented in this report are preliminary findings on the use of interval arithmetic in analysis and design of control systems. Considerable amount of work needs to be done to make it viable for applications in practical systems.

Some of the issues that were not addressed in this report and would make interesting topics for extension of this work are:

1. For the state space representation the analysis was restricted to the class of systems where the state matrix is an $\mathcal{M}$-matrix. A more general framework that addresses a broader class of systems is certainly desirable. This would encompass developing necessary and sufficient conditions for stability of less restrictive class of matrices. Several results (see list of references) in this direction have been reported, however, they tend to be computationally very demanding. Additional work is required to make these methods computationally efficient.
2. Possibilities of the use of symbolic arithmetic should be explored to reduce the conservatism experienced in analysis techniques. Symbolic arithmetic together with Global Unconstrained Optimization presented in SECTION 5.2 could provide the necessary machinery to circumvent the problem of conservatism.
3. The topic of multi-parameter as well as non-linear uncertainties can be dealt with following the approach presented in Section 5.3. Formal methodologies as well as mathematical justification needs to be investigated.

4. The results presented for state feedback design are restricted to single input single output systems. An extremely important research problem could be to explore the possibility of extending them to multi-input systems.

5. In practical control systems, state feedback may not always be feasible, hence investigation of observer based feedback as well as output feedback problems are important. One possible approach to address output feedback control may be the generalization of methods that use Sylvester equations for feedback control.

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